Chapter 1
Complex and Hypercomplex Numbers

The theory of complex numbers is well developed; complex numbers have been used in science and engineering for a long time and are still being used for solving many new problems. The arithmetic of these numbers generalizes the arithmetic of real numbers in the sense that, together with the operations of addition and multiplication by real numbers, the inverse number and the division are defined. Such a complete arithmetic exists for other numbers, which are called quaternions and octonions. Quaternions were first discovered by Hamilton in 1843 [6]. More recently, quaternions have been employed in bioinformatics, navigation systems [7], and image and video processing [8, 9]. Octonions, which are defined as doubled quaternion numbers [34], have been used in signal and image processing, and we believe that they can also be used effectively for parallel processing many images.

Recently, the theory of quaternion algebra has been used in the application of color science that processes the three color channels simultaneously [1]–[13]. Quaternion numbers found interesting applications in color image processing, such as image enhancement [19, 31], watermarking [32], adaptive filtering [33], and prostate cancer Gleason grading [16]. The quaternion can be considered as a four-dimensional number with one real part and three imaginary parts. The imaginary dimensions are represented as i, j, and k, which are orthogonal to each other and to 1. In many cases, it is useful to transfer the calculations from the real space of signals and images to complex space, analyze and solve problems by using methods of the complex analysis (arithmetic), and then transfer the solution back to the real space. The transformation to the space of quaternions is also promising. Quaternion algebra for color imaging was first used by Pei and led to the description of new tools, such as the quaternion Fourier transforms and correlations for image processing by representing the red, green, and blue values at each pixel in the color image as single pure quaternion-valued pixels [10]. There are a number of studies on quaternions and quaternion operations and systems in color image processing [11]–[19]. These color-processing systems use pure complex quaternion representation, not the complete quaternion components.

It is natural to ask how to use the complete quaternion representation, or more precisely, how to use the "real" scalar number information in different color image-processing applications, or what the advantage is of using the complete representation model over the pure complex quaternion model, particularly in color image processing. The demand by the consumer to run multiple applications on a mobile platform is increasing every day and is presenting a greater challenge to develop efficient computation tools and proficient energy resources [8]. For example, the users want their cell phones to work on different communication standards with multiple office or home-related applications showing real-time performance.

1.1 Complex and Hypercomplex Numbers

In this section, we present the basics of the complex and hypercomplex numbers, namely quaternions, and describe the main operations and properties of such numbers. Complex numbers are considered as points on a two-dimensional plane, where one coordinate is real and another one is measured along the imaginary vertical, as shown in Fig. 1.1 for the point $(3, 2i)$. The first coordinate, 3, of this point is measured along the real axis, and the unit on the vertical is denoted by $1i$, or simply i. This unit measure i is called *the imaginary unit* and will be described in the next sub-section.

1.1.1 Complex arithmetic

We first go back to the sixteenth century, when the formal and not real solution of the simple quadratic equation

$$z^2 + 1 = 0, \quad \text{or} \quad z^2 = -1, \tag{1.1}$$

was denoted by $z = \sqrt{-1}$ by two Italian mathematicians Rafael Bombelli and Gerolamo Cardano [34]. Another solution of this equation is $z = -\sqrt{-1}$.

Figure 1.2(a) shows the graph of the parabola $y = x^2 + 1$, which does not intersect the horizontal line, i.e., there is no solution of Eq. (1.1) in real arithmetic. Solutions to this equation can be found in other arithmetics that

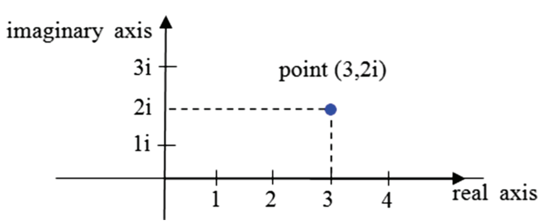

Figure 1.1 The point on a two-dimensional plane.

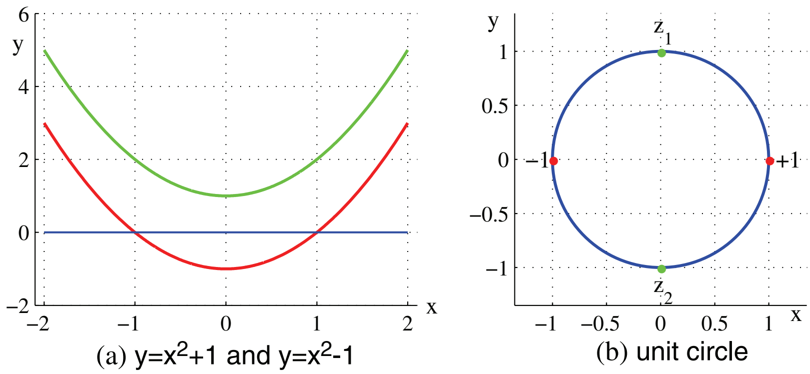

Figure 1.2 (a) Parabola and (b) the unit circle with four unit points on it.

will be described in a moment; in fact, these two solutions are on the unit circle shown in Fig. 1.2(b). There are four unit points on this circle. Two points are on the horizontal line, ± 1, and they are the solutions of the equation $z^2 - 1 = 0$, which can be written as $x^2 - 1$ with real numbers x. The graph of the parabola $y = x^2 - 1$ is also shown in Fig. 1.2(a). Two other points of the circle, which are on the vertical line, are the solutions of the equation $z^2 + 1 = 0$. These are the points $z_1 = \sqrt{-1}$ and $z_2 = -\sqrt{-1}$.

Given a real number $y \neq 0$, the solutions of the equation

$$z^2 + y^2 = 0, \quad \text{or} \quad \left(\frac{z}{y}\right)^2 + 1 = 0, \tag{1.2}$$

can be written as $z/y = \pm\sqrt{-1}$, $z = \pm y\sqrt{-1}$, or $z = \pm(\sqrt{-1})y$. If for a given real number x we consider the quadratic equation

$$(z - x)^2 + y^2 = 0, \tag{1.3}$$

then, one can see that $z - x = \pm y\sqrt{-1}$ and the solutions of this equation are the numbers $z = x + y\sqrt{-1} = x + (\sqrt{-1})y$ and $z = x - y\sqrt{-1} = x - (\sqrt{-1})y$.

In the eighteenth century, Euler denoted this imaginary number or symbol $\sqrt{-1}$ by i, i.e., $i = \sqrt{-1}$ and $i^2 = -1$. This symbol represents an imaginary unit, and in the engineering community it is denoted by j, since the letter "i" is used for the electrical current.

The number $z = x + iy = x + yi$ is called *a complex number*. Numbers x and y are real, and x is called the real part of z and y is the imaginary part of z. These two parts are denoted by $x = \mathrm{Re}\,(z)$ and $y = \mathrm{Im}\,(z)$, or $x = \mathfrak{R}(z)$ and $y = \mathfrak{I}(z)$. The concept of the complex number generalizes the real numbers that can be considered as the complex numbers with zero imaginary parts, i.e., when $y = 0$. Arithmetical operations also are generalized in complex arithmetic, and we consider the main operations over complex numbers.

Given complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the following properties for operations of addition and multiplication are valid:

1. $z_1 + z_2 = [x_1 + iy_1] + [x_2 + iy_2] = (x_1 + x_2) + i(y_1 + y_2)$,
2. $kz_1 = k[x_1 + iy_1] = (kx_1) + i(ky_1)$,

for any real number k. One can note that $z_1 + z_2 = z_2 + z_1$. The operation of subtraction is defined as $z_1 - z_2 = z_1 + (-1)z_2$:

3. $z_1 - z_2 = [x_1 + iy_1] - [x_2 + iy_2] = (x_1 - x_2) + i(y_1 - y_2)$.

The most important operation of complex numbers is multiplication, $z = z_1 z_2$, which is calculated directly as

$$z_1 z_2 = [x_1 + iy_1][x_2 + iy_2] = x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2.$$

Considering the definition, $i^2 = -1$, we obtain the following:

4. $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$.

Thus, the multiplication of two complex numbers z_1 and z_2 is the complex number $z = x + iy$ with the real and imaginary parts defined by $x = x_1 x_2 - y_1 y_2$ and $y = x_1 y_2 + y_1 x_2$, respectively. When the numbers z_1 and z_2 are real, i.e., $y_1 = 0$ and $y_2 = 0$, this operation is reduced to the multiplication of real numbers, $z_1 z_2 = x_1 x_2$. The set of complex numbers is denoted by C.

Example 1.1

If $z_1 = 1 + 2i$ and $z_2 = 2 - 3i$, then the multiplication proceeds as

$$z_1 z_2 = [2 - 2(-3)] + i[-3 + 2(2)] = 8 + i.$$

The operation of multiplication together with the operations of addition and subtraction is commutative, i.e.,

5. $z_2 z_1 = (x_2 x_1 - y_2 y_1) + i(x_2 y_1 + y_2 x_1) = z_1 z_2$.

It is not difficult to see that for any real numbers k_1 and k_2 and complex numbers z_1, z_2, and z_3, the following holds:

5a. $(k_1 z_1)(k_2 z_2) = (k_1 k_2)(z_1 z_2)$,

5b. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

For a complex number $z_1 = x_1 + iy_1$, the number $z_2 = x_1 - iy_1$ is called *the complex conjugate* and denoted by $\bar{z}_1$. It is clear that $\bar{\bar{z}}_1 = z_1$ for any complex number, and $\bar{z}_1 = z_1$, if only the number z_1 is real. The operation of conjugation $z \to \bar{z}$ is a linear operation, namely,

6. $\overline{z_1 + kz_2} = \bar{z}_1 + k\bar{z}_2,$

for any real number k and complex numbers z_1 and z_2.

The module $|z_1|$ of the complex number z_1 is defined as the multiplication $z_1\bar{z}_1$, which according to multiplication can be written as

$$6a. \quad z_1\bar{z}_1 = (x_1x_1 + y_1y_1) + i(-x_1y_1 + y_1x_1) = x_1^2 + y_1^2$$

and denoted as $|z_1|^2$. Therefore, $|z_1| = \sqrt{x_1^2 + y_1^2}$ and it is positive if $z_1 \neq 0$, and it is clear that $|z_1| \geq |x_1|$ and $|z_1| \geq |y_1|$. The number $|z_1|$ also is called *the length* of z_1.

In the general case, the following holds for the complex conjugate of the multiplication:

$$6b. \quad \overline{z_1z_2} = (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2) = \bar{z}_1\bar{z}_2.$$

The following equalities hold for multiplication:

$$|z_1z_2|^2 = (z_1z_2)\overline{(z_1z_2)} = (z_1z_2)(\bar{z}_1\bar{z}_2) = (z_1\bar{z}_1)(z_2\bar{z}_2) = |z_1|^2|z_2|^2.$$

Therefore, the length of the product of two complex numbers equals the product of their lengths:

$$6c. \quad |z_1z_2| = |z_1||z_2|.$$

Example 1.2

For the complex number $z_1 = 3 + 4i$, the length of the number is

$$|z_1| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5.$$

The complex conjugate number $\bar{z}_1 = 3 - 4i$ has the same length:

$$|\bar{z}_1| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The property of triangle inequality holds for the complex numbers

$$6d. \quad |z_1 + z_2| \leq |z_1| + |z_2|.$$

The equality holds for the cases when one of the complex numbers is zero, or $z_1 = kz_2$ when k is real.

When z_1 is an imaginary number, i.e., $x_1 = 0$, the square of the number is $z_1^2 = (iy_1)^2 = -y_1^2$ and therefore,

$$6e. \quad z_1^2 = -|z_1|^2.$$

Example 1.3

For the complex number $z_1 = 4i$, the length of the number is

$$|z_1| = \sqrt{4^2} = 4, \quad |z_1|^2 = 16, \quad \text{and} \quad z_1^2 = (4i)^2 = 16i^2 = -16.$$

From the equation $z_1\bar{z}_1 = |z_1|^2$, the inverse number $1/z_1$ is defined as

$$7. \quad (z_1)^{-1} = \frac{1}{z_1} = \frac{1}{|z_1|^2}\bar{z}_1, \quad \text{when} \quad z_1 \neq 0.$$

Indeed, one can verify that $(z_1)^{-1}(z_1) = (z_1)(z_1)^{-1} = 1$. Therefore, the operation of division of the complex numbers z_2 and z_1, if only $z_1 \neq 0$, is defined as

$$8. \quad \frac{z_2}{z_1} = z_2(z_1)^{-1} = z_2\frac{1}{z_1} = z_2\frac{1}{|z_1|^2}\bar{z}_1 = \frac{1}{|z_1|^2}z_2\bar{z}_1 = \frac{z_2\bar{z}_1}{|z_1|^2}.$$

Example 1.4

For the complex numbers $z_1 = 3 - 4i$ and $z_2 = 2 + i$, we have the following:

$$\frac{1}{z_1} = \frac{1}{3-4i} = \frac{1}{(3^2+4^2)}\overline{(3-4i)} = \frac{1}{25}(3+4i) = \frac{3}{25} + i\frac{4}{25}$$

and

$$\frac{z_2}{z_1} = \frac{2+i}{3-4i} = \frac{1}{25}(2+i)(3+4i) = \frac{1}{25}(2+11i) = \frac{2}{25} + i\frac{11}{25}.$$

In complex arithmetic [35, 36], we can solve Eqs. (1.1) to (1.3). For that, we first consider the complex numbers z on the imaginary line, i.e., $z = iy$. The solution of equation $z^2 + 1 = 0$, which for such complex numbers is $-y^2 + 1 = 0$, are $z = i(+1) = i$ and $z = i(-1) = -i$. Figure 1.3(a) shows a graph of the parabola $z^2 + 1 = (iy)^2 + 1$, when iy runs the interval of imaginary numbers $[-2i, 2i]$. One can see that this parabola intersects the horizontal at points $+i$ and $-i$. Another parabola $(z-1)^2 + 2$ is shown in Fig. 1.3(b). It should be noted that the graph of this parabola is calculated for the complex numbers $z = 1 + iy$ when the imaginary components y run the interval $[-2, 2]$. Therefore, the plot is shown versus these complex numbers $z = 1 + iy$. This parabola intersects the horizontal line at two points, $z_1 = 1 + i\sqrt{2}$ and $z_2 = 1 - i\sqrt{2}$.

1.1.1.1 Geometry of complex numbers

Every complex number $z = x + iy$ can uniquely be presented as the point (x, y) on the real plane R^2. In fact, this is another form of representing the complex number. We denote this point by $P = P(z)$, which has the Cartesian

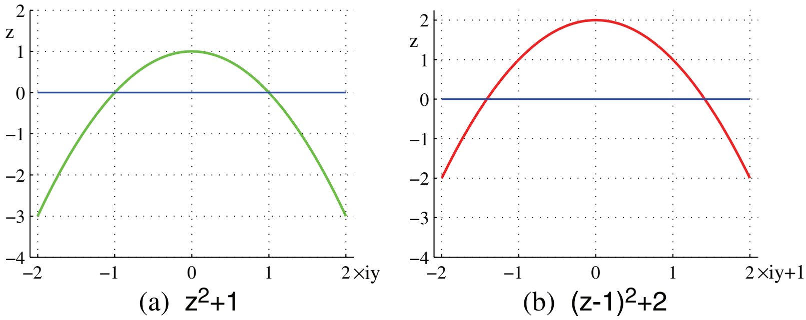

Figure 1.3 Graphs of (a) the parabola z^2+1 and (b) the parabola $(z-1)^2+2$.

coordinates (x, y). The expression $P = x + iy$ relates to complex arithmetic. The distance between the original and P is $d(P) = |z| = \sqrt{x^2 + y^2}$. So, the point P lies on the circle of radius $r = d(P)$.

In the specific case where $r = 1$, when point P lies on the unit circle, we can write P as

$$P = P(\theta) = (x, y) = (\cos\ \theta,\ \sin\ \theta) = \cos\ \theta + i\ \sin\ \theta,$$

where the imaginary unit $i^2 = -1$. The cosine and sine functions are calculated by the known Taylor series,

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \frac{\theta^{10}}{10!} + \cdots,$$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \cdots.$$

Similarly, the point $\bar{P} = P(-\theta)$, which represents the complex conjugate number $\bar{z}$, can be written as

$$\bar{P} = P(-\theta) = (x, -y) = (\cos\ \theta, -\sin\ \theta) = \cos\ \theta - i\ \sin\ \theta.$$

Both points P and $\bar{P}$ are on the same circle of radius $r = d(P) = d(\bar{P})$. Therefore,

$$P(\theta) + P(-\theta) = 2\ \cos\ \theta,$$

$$P(\theta) - P(-\theta) = i2\ \sin\ \theta.$$

The function $P(\theta)$ is denoted by $e^{i\theta}$ in honor of Euler, who used this function and founded the above relations for the real and imaginary parts of $e^{i\theta}$,

$$e^{i\theta} + e^{-i\theta} = 2\ \cos\ \theta, \tag{1.4}$$

$$e^{i\theta} - e^{-i\theta} = i2\ \sin\ \theta. \tag{1.5}$$

The point on the unit circle is completely described by the angle θ. The unit circle intersects the axes at the points

$$1 = i^0, \quad i = i^1, \quad -1 = i^2, \quad -i = (i^2)i = i^3.$$

The points $P(\theta)$ on the unit circle with $\theta = \pi, \pi/2, \pi/3$, and $\pi/4$ are the following:

$$\begin{aligned}
e^{i\pi} &= \cos(\pi) + i\,\sin(\pi) = -1 + i0 = -1,\\
e^{i\pi/2} &= \cos(\pi/2) + i\,\sin(\pi/2) = 0 + i1 = 1i = i,\\
e^{i\pi/3} &= \cos(\pi/3) + i\,\sin(\pi/3) = 0.5 + i\frac{\sqrt{3}}{2} = 0.5 + 0.8660i,\\
e^{i\pi/4} &= \cos(\pi/4) + i\,\sin(\pi/4) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = 0.7071 + 0.7071i.
\end{aligned}$$

Since $i^{2n} = (-1)^n$ and $i^{2n+1} = (-1)^n i$ for integers $n = 0, 1, 2, \ldots,$ the Taylor series for the complex exponential function $\exp(i\theta)$ can be obtained as

$$\begin{aligned}
\cos(\theta) + i\,\sin(\theta) = & 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^8}{8!} + \frac{(i\theta)^{10}}{10!} + \cdots\\
& + i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + \frac{(i\theta)^9}{9!} + \frac{(i\theta)^{11}}{11!} + \cdots =\\
e^{i\theta} = & \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}, \quad (\text{considering } 0! = 1).
\end{aligned}$$

In the general case, when point P lies on the circle of radius r, the point can be represented as (r, θ). This is the so-called *polar form* of representation of P (as shown in Fig. 1.4) instead of $P = (x, y) = (r\cos\theta, r\sin\theta)$,

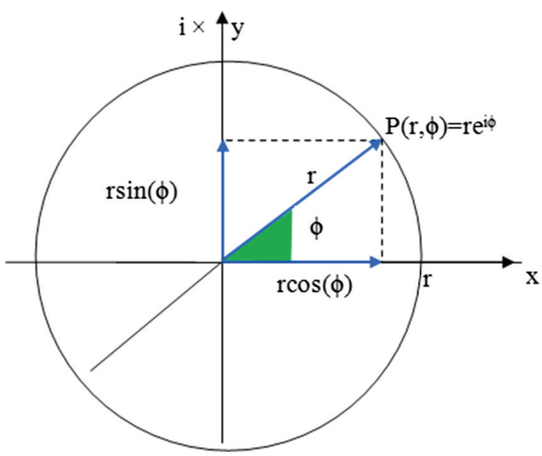

Figure 1.4 Point $P(r, \theta)$ on the complex plane C.

$$P = P(r, \theta) = re^{i\theta},$$

or we can simply write $P = (r, \theta)$. Here, the radius and angle are calculated as

$$r = \sqrt{x^2 + y^2}, \quad \text{and} \quad \theta = \arctan\frac{y}{x} = \tan^{-1}\frac{y}{x},$$

and $\theta = \arccos(y) = \pm\pi/2$, if $x = 0$. This angle θ is also called the argument of z and denoted by $\theta = \arg(z)$. If point P lies in the left semi-plane, calculation of angle θ should be corrected by adding angle $\pm\pi$ depending on the sign of the imaginary part y.

Example 1.5

The complex number $z = 3 + 4i$ as the point $P(z) = (3, 4)$ is described in the polar form as $(r, \theta) = (5, 0.9273)$. Indeed,

$$r = \sqrt{3^2 + 4^2} = 5 \quad \text{and} \quad \theta = \arctan\frac{4}{3} = 0.9273 \text{ rad},$$

or

$$\theta = \frac{180}{\pi} \times \arctan\frac{4}{3} = 53.1301°.$$

The polar form for the complex conjugate $\bar{z} = 3 - 4i$ is $P(\bar{z}) = (5, -0.9273)$,

$$r = \sqrt{3^2 + (-4)^2} = 5, \quad \text{and} \quad \theta = \arctan\frac{-4}{3} = -0.9273 \text{ rad},$$

or $\theta = -53.1301°$.

Now, we describe the geometry of the multiplication of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Considering these numbers as two-dimension vectors

$$z_1 = (x_1, y_1)^T = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \text{and} \quad z_2 = (x_2, y_2)^T = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

the complex number $z = x + iy = z_1 z_2$ can be calculated in vector form as

$$z = \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

The determinant of the matrix $\mathbf{A}$ is $\det(\mathbf{A}) = x_1^2 + y_1^2$ and we consider the case when $\det(\mathbf{A}) \neq 0$. This matrix multiplication also can be written as

$$z = \begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{x_1^2 + y_1^2}\begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+y_1^2}} & -\frac{y_1}{\sqrt{x_1^2+y_1^2}} \\ \frac{y_1}{\sqrt{x_1^2+y_1^2}} & \frac{x_1}{\sqrt{x_1^2+y_1^2}} \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

or

$$z = \begin{pmatrix} x \\ y \end{pmatrix} = |z_1| \begin{pmatrix} \cos\,\theta_1 & -\sin\,\theta_1 \\ \sin\,\theta_1 & \cos\,\theta_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Angle θ_1 is defined from the equations

$$\cos\,\theta_1 = \frac{x_1}{\sqrt{x_1^2 + y_1^2}} \quad \text{and} \quad \sin\,\theta_1 = \frac{y_1}{\sqrt{x_1^2 + y_1^2}}.$$

The matrix

$$\mathbf{A} = A(r_1, \theta_1) = r_1 \begin{pmatrix} \cos\,\theta_1 & -\sin\,\theta_1 \\ \sin\,\theta_1 & \cos\,\theta_1 \end{pmatrix}$$

is the matrix of rotation by angle θ_1 around the circle of radius amplified by a factor of $r_1 = |z_1|$. Thus, when multiplying the complex number z_2 by z_1, the point (x_2, y_2) is rotated by angle θ_1 on a circle of radius $r_2 = |z_2|$ and then is moved along its angle to another circle of radius $r_1 r_2$. The result of multiplication is the same if point (x_2, y_2) is first moved along angle $\theta_2 = \tan^{-1}(y_2/x_2)$ to the circle of radius $r_1 r_2$ and then rotated by angle θ_1 on this new circle.

Example 1.6

Consider the multiplication of two complex numbers $z_1 = 2 + i$ and $z_2 = 3 - 4i$. The product $z = z_1 z_2 = (2 \cdot 3 + 1 \cdot 4) + i(-4 \cdot 2 + 1 \cdot 3) = 10 - 5i$. In this case, $r_1 = \sqrt{2^2 + 1} = \sqrt{5} = 2.2360$, and angle $\theta_1 = 0.4636$ rad (or 26.5651°). The second complex number is on the circle of radius $r_2 = \sqrt{3^2 + 4^2} = 5$ along the ray at angle $\theta_2 = -0.9273$ rad (or $-53.1301°$) to the horizontal axis. The two points representing complex numbers z_1 and z_2 are shown in Fig. 1.5(a).

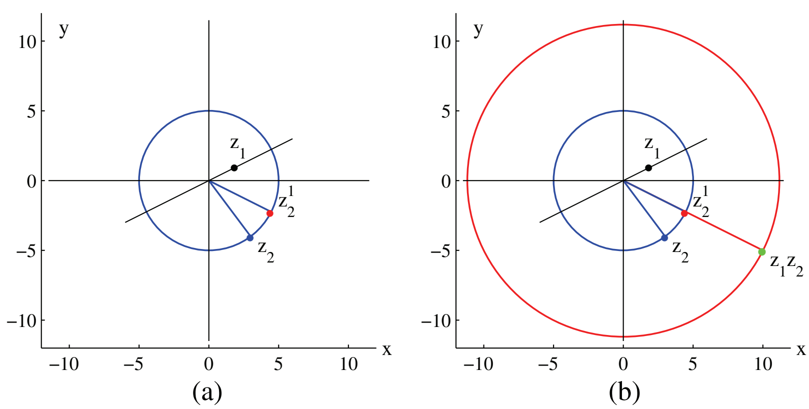

Figure 1.5 Geometry of multiplication of two complex numbers: (a) rotation of the point z_2 and (b) moving to another circle.

The second point with the coordinate $(3, -4)$ is rotated counter-clockwise to the point $(4.4720, -2.2360)$, which is denoted by z_2' in the figure and calculated by

$$\begin{pmatrix} 4.4720 \\ -2.2360 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 0.8944 & -0.4472 \\ 0.4472 & 0.8944 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

Then, this new point is moved along the angle $\theta = \theta_1 + \theta_2 = -0.4636$ rad (or $-26.5651°$) to the new circle of radius $r = r_1 r_2 = 11.1803$, by multiplying the coordinates of the point by $r_1 = \sqrt{5}$, as shown in Fig. 1.5(b). This new point can be calculated as

$$P(z_1 z_2) = \sqrt{5}\begin{pmatrix} 4.4720 \\ -2.2360 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \end{pmatrix}.$$

Therefore, $z_1 z_2 = 10 - 5i$ and, indeed, $r = |z_1 z_2| = \sqrt{10^2 + 5^2} = 11.1803$. All numbers in the above calculations are given with a precision of four digits after the point.

Example 1.7

Consider the multiplication of the complex number $z_2 = x_2 + iy_2$ by the complex number $z_1 = x_1 + iy_1 = 1 + i$. In this case, $x_1 = y_1 = 1$ and the complex number $z = z_1 z_2 = x + iy$ can be calculated as

$$z = \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

with the determinant of the matrix $\mathbf{A}$ equal to $\det(\mathbf{A}) = 2x_1 y_1 = 2$. This binary matrix $\mathbf{A}$ is well-known as a basic 2×2 matrix of the Hadamard and Fourier transformations. In this case, radius $r_1 = \sqrt{2}$ and angle $\theta_1 = \arctan(1) = \pi/4$. The matrix

$$\mathbf{A} = A(\sqrt{2}, \pi/4) = \sqrt{2}\begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}$$

describes the rotation of point (x_2, y_2) by an angle of $\pi/4$ (in radians) following the translation of the point from the circle of radius $r_2 = \sqrt{x_2^2 + y_2^2}$ to the circle of radius $\sqrt{2} r_2$.

It is not difficult to see that the complex number $z = x + iy = z_1 z_2$ can also be calculated in vector form with another matrix,

$$z = \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

Here, matrix $\mathbf{A}$ is defined by the second complex number z_2 and can be written as

$$\mathbf{A} = A(r_2, \theta_2) = r_2 \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix},$$

where radius $r_2 = \sqrt{x_2^2 + y_2^2}$ and the angle θ_2 of rotation of point (x_1, y_1) is defined from the equations

$$\cos\theta_2 = \frac{x_2}{\sqrt{x_2^2 + y_2^2}} \quad \text{and} \quad \sin\theta_2 = \frac{y_2}{\sqrt{x_2^2 + y_2^2}}.$$

In the multiplication $z_1 z_2$ by this matrix, point (x_1, y_1) can be rotated by angle θ_2 on a circle of radius $r_1 = |z_1|$ and then moved to another circle of radius $r_2 r_1$.

1.1.2 Quaternion numbers

The operation of multiplication in complex arithmetic allows for division by nonzero numbers. As shown in the previous sub-section, division is possible because of the unique inverse complex number $1/z$, when $z \neq 0$. If we try to generalize the concept of complex numbers and define a new arithmetic, the operation of multiplication must guarantee the existence of such inverse numbers. The operations of multiplication and division are dual, meaning that for any two complex numbers $z_1, z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{1}{\frac{1}{z_1} z_2}.$$

Let C be the complex plane with numbers that we denote by $z = a + ib$, where a and b are real numbers, and i is the imaginary unit. We may consider a different imaginary unit j and new numbers q in the form of

$$q = z + jc = (a + ib) + jc = a + bi + jc,$$

where c is real. It is well known that for such numbers with three components (a, b, c), it is not possible to define the multiplication together with division. For example, consider the simple imaginary number $q = i = 0 + 1i + 0j$ and its inverse $q^{-1} = (a + ib + jc)$, if it exists, from the equation

$$(0 + 1i + 0j)(a + ib + jc) = 1.$$

We can write $1 = i(a + ib + jc) = ia + i^2 b + ijc = ia - b + ijc$, or $ia + ijc = 1 + b$ and, therefore, $a = 0$ and $ijc = 1 + b$. If we multiply both parts of this equality by i from the left, we obtain $i(ijc) = i(1 + b)$, or $-jc = i(1 + b)$. This is possible only if $j = -i(1 + b)/c$, but j should be a different imaginary unit, not proportional to i. Thus, the inverse number to i does not exist in the arithmetic of numbers $a + ib + jc$.

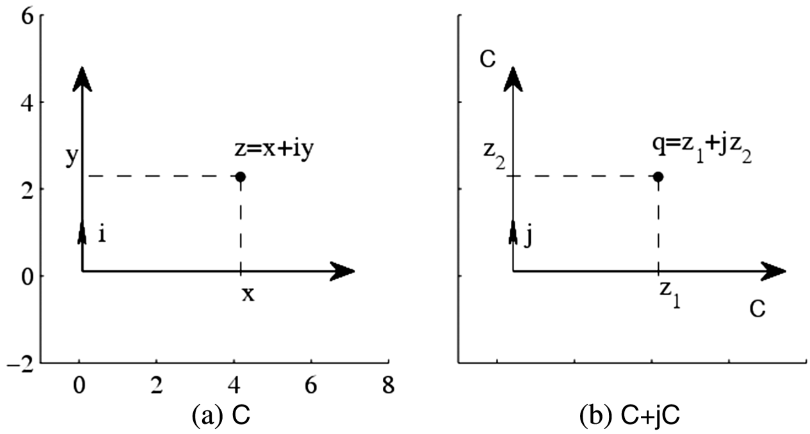

Figure 1.6 (a) The complex plane C and (b) the abstract combination of two complex spaces.

The complex numbers $z = x + iy$ represent the points (x, y) in the complex space C. Figure 1.6(a) shows such a plane. As shown above, adding one line to the complex plane does not result in a full arithmetic with multiplication and division. Instead, we may think of "adding" similar complex planes, formally writing this operation as $C + jC$, as shown in Fig. 1.6(b). For that, we can imagine two complex planes; one complex plane C with numbers $z_1 = x_1 + iy_1$ and another complex plane C with numbers $z_2 = x_2 + iy_2$. If we assume another imaginary unit j, then the following numbers can be considered:

$$q = z_1 + jz_2 = (x_1 + iy_1) + j(x_2 + iy_2). \tag{1.6}$$

We can denote such a doubled complex space by C^2 and call such representation of numbers q *the* $(2, 2)$*-representation.*

Thus, the complex numbers z_1 and z_2 in this construction play the same role as the real numbers x and y in the complex numbers $x + iy$. These new numbers q can be written as

$$q = x_1 + iy_1 + jx_2 + (ji)y_2,$$

where the number (ji) should represent another number, or may be a new imaginary number unit, which will be denoted by k or $-k$, i.e., $k = ji$ or $-k = ji$. These numbers as elements of four (or "quaternion" in Latin) are called *quaternions* and were first described by the Irish mathematician William Hamilton (1805–1865) in Ref. [6].

1.1.3 Rules for multiplication

The new imaginary number j has been considered, and a new number k has been defined as $k = ji$. Now, we need to describe the multiplications among the numbers i, j, and k. We assume that the operation of multiplication of numbers q is associative, i.e., $(q_1q_2)q_3 = q_1(q_2q_3)$, for any numbers q_1, q_2, and q_3 of type

$$q = x_1 + iy_1 + jx_2 + ky_2. \tag{1.7}$$

Also we assume that all such numbers q are commutative with real numbers λ, i.e., $q\lambda = \lambda q$.

It is important to note that the multiplication of such numbers q is not commutative, i.e., $q_1q_2 \neq q_1q_2$, for many quaternions where $q_1 \neq q_2$. Indeed, we consider the number $q = 1 + i + j$. Then, the following calculations hold:

$$\begin{aligned} qi - qj &= (1 + i + j)i - (1 + i + j)j = (i + i^2 + ji) - (j + ij + j^2) \\ &= (i - 1 + ji) - (j + ij - 1) = i - j + ji - ij, \end{aligned}$$

or

$$q(i - j) = (i - j) + (ji - ij).$$

If $ij = ji$, then $q = 1$, which is not true. Thus, $ij \neq ji$, and it can similarly be shown that $jk \neq kj$ and $ik \neq ki$.

The following properties can also easily be obtained:

Since $j^2 = -1$,

$$\begin{aligned} jk &= j(ji) = (j^2)i = -i, \\ jki &= j(ji)i = (jj)ii = -i^2 = 1, \\ ijk &= ij(ji) = i(j^2)i = -i^2 = 1, \\ ki &= (ji)i = j(i^2) = -j. \end{aligned}$$

It is also true that

$$(ki)^2 = -1 \quad \text{and} \quad (kj)^2 = -1,$$

since

$$\begin{aligned} (ki)(ki) &= (ji)i(ji)i = j(ii)j(ii) = j(-1)j(-1) = jj = -1 \\ (kj)(kj) &= (ji)j(ji)j = (ji)jj(ij) = -(ji)(ij) = -j(ii)j = jj = -1. \end{aligned}$$

It is clear that these equalities do not describe all rules of multiplication from the left and right between the numbers i, j, and k. Since we have not yet defined the multiplication ij, this system of equalities is incomplete; other missing multiplications are ik, kj, and k^2.

We know that $ij \neq ji$, therefore, we can assume the following condition:

$$ij = -ji, \quad \text{or} \quad ij = -k.$$

Then, we obtain the following equalities:

$$\begin{aligned}
k^2 &= kk = (ji)(ji) = (-ij)(ji) = -i(jj)i = ii = -1,\\
kj &= (ji)j = -(ij)j = -i(jj) = i,\\
ik &= i(ji) = i(-ij) = -(ii)j = j.
\end{aligned}$$

It is also true that one of these equalities results in the equality $ij = -ji$. For instance, let $k^2 = -1$; then we obtain the following:

$$[jiji = -1] \to [j \cdot jiji = -j] \to [(jj)iji = -j] \to [iji = j] \to [-ji = ij].$$

Thus, under the assumption that $ij = -ji$, we obtain the following full set of multiplication laws:

$$\begin{aligned}
ij &= -ji = -k, \quad jk = -kj = -i,\\
ki &= -ik = -j, \quad i^2 = j^2 = k^2 = kji = jik = -1.
\end{aligned}$$

Together with multiplication by the real unit number 1, all of these basic multiplications are shown in Table 1.1, which is referred to as the $T(1,j,i,k)$ table; the order of imaginary units j, i, k shows that if we periodically extend this triplet as j, i, k, j, i, then the result of multiplication of each of the two numbers from left to right (right to left) is the next number with the plus (minus) sign.

It should be noted that such a table of multiplications is not unique, and we can use other tables. For instance, if we start with the condition $ji = -k$, we obtain the similar table $T(1,j,i,k)$, which is shown in Table 1.2.

Table 1.1 Table $T(1,j,i,k)$ of the multiplications of the four quaternion unit numbers.

	1	i	j	k
1	1	i	j	k
i	i	-1	$-k$	j
j	j	k	-1	$-i$
k	k	$-j$	i	-1

Table 1.2 Table $T(1,i,j,k)$ of the multiplications of the four quaternion unit numbers

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

The three imaginary units i, j, and k satisfy the following multiplication laws:

$$\begin{aligned} ij &= -ji = k, \quad jk = -kj = i, \\ ki &= -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1. \end{aligned} \tag{1.8}$$

In these two tables, the unit i is changed by j and j is changed by i. We will stand on $T(1, i, j, k)$ since this table is most often used in quaternion arithmetic. The multiplication of quaternion numbers, which is based on this table of rules, is called *the left multiplication* of quaternions.

1.1.4 Basic operations

We consider the arithmetical operations with quaternion numbers. Thus, a quaternion number is represented as

$$q = a + bi + cj + dk = a + (bi + cj + dk),$$

where a, b, c, and d are real numbers. The number a is called the real part of q, and $(bi + cj + dk)$ is the "imaginary" part of q. The quaternion q has a three-component imaginary part, which we denote by q', such that

$$q = a + q' = a + (bi + cj + dk) = a + (ib + jc + kd).$$

Other notations for the real and imaginary components are Sq and Vq, respectively, i.e., $Sq = a$, $Vq = q'$, and $q = Sq + Vq$.

In the cases when $q' = ib, jc$, and kd, the numbers $q = a + ib$, $q = a + jc$, and $q = a + kd$ can be considered as complex numbers. The zero quaternion is $0 + 0i + 0j + 0k = 0$, and the unity element is $1 + 0i + 0j + 0k = 1$.

Now, we describe the arithmetic of quaternion numbers, i.e., the main operations that include the sum, multiplication by a real constant, and multiplication and division of quaternion numbers.

The sum of two quaternions $q_1 = a_1 + q'_1 = a_1 + (ib_1 + jc_1 + kd_1)$ and $q_2 = a_2 + q'_2 = a_2 + (ib_2 + jc_2 + kd_2)$ is defined as

1. $q = q_1 + q_2 = a + q' = (a_1 + a_2) + (q'_1 + q'_2) = a + (bi + cj + dk),$

and $b = b_1 + b_2$, $c = c_1 + c_2$, and $d = d_1 + d_2$. Therefore, this operation is defined to be component-wise and commutative:

1*a*. $q = q_1 + q_2 = q_2 + q_1.$

For any real number λ,

2. $\lambda q = \lambda a + \lambda q' = \lambda a + (i\lambda b + j\lambda c + k\lambda d).$

If $a = 0$, the quaternion q is called a pure quaternion number:

$$q = 0 + (ib + jc + kd).$$

The multiplication q_1q_2 of two quaternions q_1 and q_2 is calculated according to Table 1.2 of multiplication as

3. $q = q_1q_2 = (a_1 + q_1')(a_2 + q_2') = a_1a_2 + a_1q_2' + a_2q_1' + q_1'q_2'$

and can be expanded as

$$\begin{aligned} q &= a_1a_2 + a_1q_2' + a_2q_1' + (ib_1 + jc_1 + kd_1)(ib_2 + jc_2 + kd_2) \\ &= a_1q_2' + a_2q_1' + a_1a_2 - (b_1b_2 + c_1c_2 + d_1d_2) \\ &\quad + i(c_1d_2 - d_1c_2) - j(b_1d_2 - d_1b_2) + k(b_1c_2 - c_1b_2). \end{aligned}$$

This multiplication $q = q_1q_2$ can also be written as

$$q = [a_1q_2' + a_2q_1'] + a_1a_2 - (b_1b_2 + c_1c_2 + d_1d_2) + \begin{vmatrix} i & j & k \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix}. \qquad (1.9)$$

Example 1.8

Consider two quaternions $q_1 = 1 - 2i + 3j + k$ and $q_2 = 1 - i + 4j + 3k$. The multiplication $q = q_1q_2$ is calculated as follows.

(1). The imaginary components of these numbers are $q_1' = -2i + 3j + k$ and $q_2' = -i + 4j + 3k$; therefore,

$$a_1q_2' + a_2q_1' = 1 \cdot (-2i + 3j + k) + 1 \cdot (-i + 4j + 3k) = -3i + 7j + 4k.$$

(2). The real part of the multiplication is

$$a_1a_2 - (b_1b_2 + c_1c_2 + d_1d_2) = 1 - [-2(-1) + 3(4) + 1(3)] = 1 - 17 = -16.$$

(3). The quaternion determinant equals

$$\begin{vmatrix} i & j & k \\ -2 & 3 & 1 \\ -1 & 4 & 3 \end{vmatrix} = i(9 - 4) - j(-6 + 1) + k(-8 + 3) = 5i + 5j - 5k.$$

(4). Therefore, the quaternion number $q = q_1q_2$ equals

$$q = q_1q_2 = (-3i + 7j + 4k) - 16 + (5i + 5j - 5k) = -16 + (-2i + 12j - k).$$

It is not difficult to check that the multiplication of these two numbers is not commutative:

$$q_1q_2 \neq q_2q_1 = -16 + (-8i + 2j + 9k).$$

Below is the script of the MATLAB®-based code "test_example.m" for this example. It uses function "mult2qs_direct" to multiply two quaternions.

```
%-----------------------------------------------------
% test_example.m / with function mult2qs_direct(q1,q2)
% multiplication of two quaternion numbers
   q1= [1,−2,3,1]; q2=[1,−1,4,3];
   q=mult2qs_direct(q1,q2);     % −16 2 12 −1
%-----------------------------------------------------
 function q = mult2qs_direct(q1,q2)
   a1=q1(1); b1=q1(2); c1=q1(3); d1=q1(4);
   a2=q2(1); b2=q2(2); c2=q2(3); d2=q2(4);
   Q1=q1(2:4); Q2=q2(2:4);  % imaginary parts
%
   Q12=a1*Q2 + a2*Q1;        % −3 7 4
   a=a1*a2-Q1*Q2';           % 1−17= −16
   bi= (c1*d2−d1*c2);        % 5
   cj=-(b1*d2−d1*b2);        % 5
   dk= (b1*c2−c1*b2);        % −5
%
   b=Q12(1) + bi;            % 2
   c=Q12(2) + cj;            % 12
   d=Q12(3) + dk;            % −1
   q=[a,b,c,d];              % −16 2 12 −1
%-----------------------------------------------------
```

4. When q_1 and q_2 are the quaternions with only one and the same imaginary units, for instance, $q_1 = a_1 + jc_1$ and $q_2 = a_2 + jc_2$, the operation of multiplication q_1q_2 is complex multiplication. Indeed, the following holds:

$$q_1' = jc_1, q_2' = jc_2, \quad q_1' \times q_1' = \begin{vmatrix} i & j & k \\ 0 & c_1 & 0 \\ 0 & c_2 & 0 \end{vmatrix} = 0,$$

and

$$q_1q_2 = [a_1(jc_2) + a_2(jc_1)] + a_1a_2 - c_1c_2 = (a_1a_2 - c_1c_2) + j(a_1c_2 + a_2c_1).$$

If $q_1' = 0$ and $q_2' = 0$, then

5. $q_1q_2 = (a_1 + q_1')(a_2 + q_2') = a_1a_2.$

For any real numbers λ_1 and λ_2,

6. $(\lambda_1 q_1)(\lambda_2 q_2) = \lambda_1\lambda_2(q_1 q_2)$.

The quaternion multiplication is distributive:

7. $q_1(q_2 + q_3) = q_1q_2 + q_1q_3 \quad \text{and} \quad (q_1 + q_2)q_3 = q_1q_3 + q_2q_3,$

and associative by multiplication:

8. $(q_1q_2)q_3 = q_1(q_2q_3)$.

1.1.5 Properties of multiplication of quaternions

In this section we consider a few properties of the operation of multiplication in quaternion arithmetic.

P1. It is important to mention that the property of commutativity does not hold in quaternion algebra; i.e., for quaternions $q_1 = a_1 + q_1'$ and $q_2 = a_2 + q_2'$,

$$q_1q_2 = q_2q_1 \quad \text{or} \quad q_1q_2 \neq q_2q_1,$$

because of the imaginary units, $ij = -k \neq ji = k$, and so on. It can also be seen from Eq. (1.9) that

$$q_2q_1 = [a_2q_1' + a_1q_2'] + a_2a_1 - (b_2b_1 + c_2c_1 + d_2d_1) + \begin{vmatrix} i & j & k \\ b_2 & c_2 & d_2 \\ b_1 & c_1 & d_1 \end{vmatrix}$$

and therefore,

$$q_1q_2 - q_2q_1 = 2\begin{vmatrix} i & j & k \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix}. \tag{1.10}$$

The multiplication of a quaternion number q_1 by a real number λ is commutative, i.e., $\lambda q_1 = q_1\lambda$.

When the quaternion vectors $\mathbf{q}_1'$ and $\mathbf{q}_2'$ are similar, i.e., $\mathbf{q}_2' = \lambda\mathbf{q}_1'$, where λ is a real number, then

$$q_2q_1 = q_1q_2 = (a_2 + a_1\lambda)q_1' + a_2a_1 - \lambda(b_1^2 + c_1^2 + d_1^2).$$

P2. In the case when $q_1 = q_2$, we obtain the following for the square of the quaternion:

$$q = q_1^2 = [a_1^2 - (b_1^2 + c_1^2 + d_1^2)] + 2a_1q_1'. \tag{1.11}$$

The real part of the square is the number $a = a_1^2 - (b_1^2 + c_1^2 + d_1^2)$, and the imaginary part is $2a_1q_1'$. If q_1 is a pure quaternion, i.e., $a_1 = 0$, then

$$q_1^2 = (q_1')^2 = -(b_1^2 + c_1^2 + d_1^2) \leq 0. \tag{1.12}$$

Thus, for a quaternion number $q = a + q' = a + (ib + jc + kd)$,

$$q^2 = a^2 - |q'|^2 + 2aq', \tag{1.13}$$

and therefore, the equations of type $q^2 = 4 > 0$ cannot be solved, if $q' \neq 0$. The equations of type $q^2 = -4 < 0$ can be solved in quaternion numbers, since in this case $a = 0$ and we have the condition $q^2 = -|q'|^2 = -4$, or $|q'|^2 = 4$. The number $q = q'$ as a point (b, c, d) is on the sphere of radius 2 in three-dimensional (3-D) space. Such points on this sphere are, for instance, $(2, 0, 0), (0, 2, 0), (0, 0, -2), (\sqrt{2}, \sqrt{2}, 0)$, and $(1, 1, \sqrt{2})$. In other words, the solutions of the equation $q^2 = -4$ include the quaternion numbers $q = 2i + 0j + 0k$, $0i + 2j + 0k$, $0i + 0j - 2k$, $\sqrt{2}(i + j) + 0k$, and $i + j + \sqrt{2}k$.

Example 1.9

Consider the solution of the equation

$$q_1^2 = 1 + 2i + 4j.$$

It directly follows from Eq. (1.11) that the following system of equations should be considered:

$$\begin{cases} a_1^2 - (b_1^2 + c_1^2 + d_1^2) = 1, \\ a_1 q_1' = i + 2j. \end{cases}$$

Therefore, $d_1 = 0$, and this system of equations is reduced to the following one:

$$\begin{cases} a_1^2 - (b_1^2 + c_1^2) = 1, \\ a_1 b_1 = 1, \\ a_1 c_1 = 2, \end{cases}$$

and, therefore, $q = a_1 + (1/a_1)i + (2/a_1)j$. To find a_1, we need to solve the equation

$$a_1^2 - \left(\frac{1}{a_1^2} + \frac{4}{a_1^2}\right) = 1.$$

Two solutions of this equation $(a_1^2)^2 - (a_1^2) - 5 = 0$ are

$$a_1^2 = \frac{1 \pm \sqrt{1 + 20}}{2} = \frac{1 \pm \sqrt{21}}{2},$$

and since we are looking for real a_1, we obtain

$$a_1^2 = \frac{1+\sqrt{21}}{2}, \quad a_1 = \pm\sqrt{\frac{1+\sqrt{21}}{2}}.$$

Thus, the solutions of the above equation are

$$\begin{aligned} q &= a_1 + \frac{i}{a_1} + \frac{2j}{a_1} = a_1 + \frac{1}{a_1}(i+2j) \\ &= \pm\left[\sqrt{\frac{1+\sqrt{21}}{2}} + \frac{\sqrt{2}}{1+\sqrt{21}}(i+2j)\right] = \pm[1.6707 + 0.5985(i+2j)]. \end{aligned}$$

If the square q_1^2 is a real number, then $2a_1q_1' = 0$, which means that $a_1 = 0$ or $q_1' = 0$. In the first case, the square is negative:

$$q = q_1^2 = -(b_1^2 + c_1^2 + d_1^2) < 0.$$

The square of any pure quaternion is a negative number. In the second case, we have $q = a_1^2 > 0$. The square of the quaternion number is positive only if the number is real.

Example 1.10

Consider the first equation of this chapter in quaternion arithmetic:

$$q^2 + 1 = 0. \tag{1.14}$$

For a solution $q = a + q' = a + (ib + jc + kd)$ of this equation, we obtain the following:

$$a^2 - |q'|^2 + 1 + 2aq' = 0,$$

where $|q'|^2 = b^2 + c^2 + d^2$. The imaginary part of the expression of this equation is zero; therefore, $a = 0$ and the equation is reduced to the equation $|q'|^2 = 1$. All solutions of this equation $q = q' = (ib + jc + kd)$ as points (b, c, d) are on the unit sphere $b^2 + c^2 + d^2 = 1$. Thus, the solutions of Eq. (1.14) are pure unit quaternion numbers. The case when $c = d = 0$ corresponds to the complex solutions $q = i$ and $q = -i$.

The solutions of the general equation

$$q^2 + y^2 = 0, \tag{1.15}$$

where y is a positive number, can be obtained from the solutions of Eq. (1.14),

$$\left(\frac{q}{y}\right)^2 + 1 = 0.$$

Thus, $q/y = ib + jc + kd$, where $b^2 + c^2 + d^2 = 1$, or we can write $q = iyb + jyc + kyd$. The solutions q are pure quaternions with length $|q| = y$. Presenting such quaternions as points (yb, yc, yd), they are in the sphere of radius y.

Example 1.11

Consider all solutions $q = a + q'$ of the equation

$$q^2 - q + 2 = 0. \tag{1.16}$$

This equation can be written as

$$[2aq' + a^2 + (q')^2] - (a + q') + 2 = 0,$$

or

$$[a^2 - a + (q')^2 + 2] + q'(2a - 1) = 0,$$

and it can reduced to the following system of equations:

$$\begin{cases} q'(2a - 1) = 0, \\ a^2 - a + (q')^2 + 2 = 0. \end{cases}$$

If $q' = 0$, then we do not get a real solution from the equation $a^2 - a + 2 = 0$. Therefore, we consider the $a = 1/2$ case, for which we obtain the condition

$$b^2 + c^2 + d^2 = -(q')^2 = a^2 - a + 2 = \frac{1}{4} - \frac{1}{2} + 2 = \frac{7}{4}.$$

Thus, a solution q in Eq. (1.16) is of the form $q = 1/2 + q'$, where point $q' = (b, c, d)$ is on a sphere of radius $r = \sqrt{7}/2$. Such solutions are, for instance,

$$q_1 = \frac{1}{2} + i\frac{\sqrt{7}}{2},\ q_2 = \frac{1}{2} - j\frac{\sqrt{7}}{2},\ q_3 = \frac{1}{2} - i + j\frac{\sqrt{2}}{2} + k\frac{1}{2},\ q_4 = \frac{1}{2} + (i - j)\frac{\sqrt{2}}{2} + k\frac{\sqrt{3}}{2}.$$

The numbers q_1 and q_2 are only solutions of the equation in complex arithmetic, and q_3 and q_4 are full quaternion numbers. A sphere with the points marked q_2, q_3, and q_4 is shown in Fig. 1.7. This sphere contains all solutions of the equation $q^2 - q + 2 = 0$, which are pure quaternions.

Example 1.12 *Equation of the golden ratio*

Consider all solutions $q = a + q'$ of the following equation:

$$q^2 - q - 1 = 0. \tag{1.17}$$

This equation can be written as

$$[a^2 - a + (q')^2 - 1] + q'(2a - 1) = 0,$$

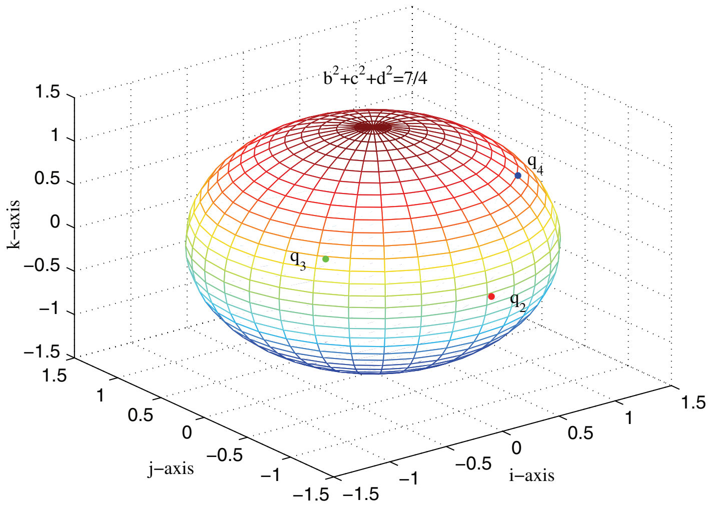

Figure 1.7 A sphere of radius $\sqrt{7}/2$ in the space of quaternion vectors.

and it can reduced to the following system of equations:

$$\begin{cases} q'(2a-1)=0, \\ a^2-a+(q')^2-1=0. \end{cases}$$

We consider two cases, when $q'=0$ and when $q'\neq 0$.

In the first case, we obtain two real solutions of the equation $a^2-a-1=0$:

$$a_1=\Phi=\frac{1+\sqrt{5}}{2} \quad \text{and} \quad a_2=\varphi=\frac{1-\sqrt{5}}{2}.$$

The number $\Phi = 1.6180339887498948\ldots$ is known as *the golden ratio*, or *the golden mean*. The second solution is $\varphi = 1-\Phi = -0.6180339887498948\ldots.$

Figure 1.8 illustrates these two solutions as the points of intersection of the line $y=x+1$ with the parabola $y=x^2$. The first point (from the left) of the intersection is $\varphi=1-\Phi$, and the second point is Φ.

In the second case when $a=1/2$, we obtain the following "condition" for the imaginary part of q to be considered:

$$|q'|^2=-(q')^2=a^2-a-1=\frac{1}{4}-\frac{1}{2}-1=-\frac{5}{4}.$$

A length of the vector cannot be negative.

Thus, in 3-D space of pure quaternions, the equation $q^2-q-1=0$ is unsolvable. The only solutions of an equation such as $a^2-a-1=0$ are in the real line, and these solutions are related to the golden ratio. This example

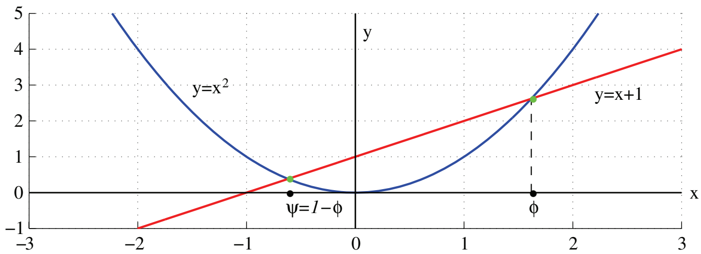

Figure 1.8 The line $y = x + 1$ and the parabola $y = x^2$ in the real plane R^2.

differs from Example 1.11, where the consideration of the equation in quaternion arithmetic results in not only known complex solutions but the entire sphere of radius $\sqrt{7}/2$. The golden ratio equation is unique in the sense that it is solvable only in the real line.

As is well known, the golden ratio had originally been defined as a ratio of two real numbers b and a (when $b > a$), such that (see Fig. 1.9)

$$\frac{b+a}{b} = \frac{b}{a}, \tag{1.18}$$

or $(b+a)b^{-1} = ba^{-1}$. Therefore, we can write this condition as $1 + ab^{-1} = ba^{-1}$, or $ba^{-1} - ab^{-1} = 1$. After denoting $x = ba^{-1}$, we obtain the equation $x - x^{-1} = 1$, or $x^2 - x - 1 = 0$.

In quaternion arithmetic, the ratios $(b+a)/b$ and b/a in Eq. (1.18) can be considered as divisions from the left or divisions from the right. In other words, instead of one equation in the real case, we have two equations for quaternion numbers,

$$(b+a)b^{-1} = ba^{-1} \quad \text{or} \quad ba^{-1} - ab^{-1} = 1,$$
$$b^{-1}(b+a) = a^{-1}b, \quad \text{or} \quad a^{-1}b - b^{-1}a = 1,$$

which are different, since the operation of multiplication of quaternions is not commutative. In both cases, these equations can be reduced to $q^2 - q - 1 = 0$, when $q = ba^{-1}$ and $a^{-1}b$, respectively.

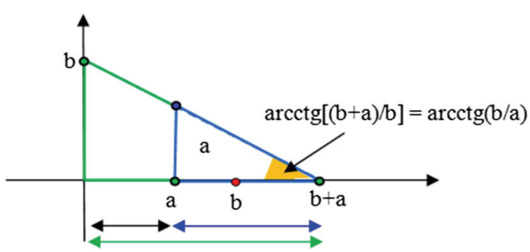

Figure 1.9 Triangles with the golden ratio property.

We consider the two quaternions $q_1 = a_1 + ib_1 + jc_1 + kd_1$ and $q_2 = a_2 + ib_2 + jc_2 + kd_2$ to be in the golden ratio if all components of these numbers have the golden ratio. Namely, let $q_2 > q_1$, meaning that $a_2 > a_1$, $b_2 > b_1$, $c_2 > c_1$, and $d_2 > d_1$. We assume the golden ratio for the components

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} = \frac{d_2}{d_1} = \Phi.$$

Then,

$$\frac{q_2 + q_1}{q_2} = \frac{(1+\Phi)q_1}{\Phi q_1} = \frac{(1+\Phi)}{\Phi} = \Phi = \frac{q_2}{q_1},$$

since $1 + \Phi = \Phi^2$.

It should be noted that the golden ratio was first introduced for the real numbers a and b, and the ratio itself concerns the ratio of the lengths $a + b$, b, and a, which are real and positive. This means that we should apply this concept to the lengths of the numbers. The quaternion number $q = a + q'$ has the real and imaginary parts with lengths $|a|$ and $|q'|$, respectively. The length of the quaternion is $|q|$, and we know that $|q| \leq |a| + |q'|$. Let us assume that $|a| < |q'|$ and these two parts are in the golden ratio, i.e., $|q'| = \Phi|a|$. Then,

$$|q| = \sqrt{a^2 + |q'|^2} = \sqrt{(1+\Phi^2)a^2} = |a|\sqrt{1+\Phi^2} = |a|\sqrt{2+\Phi}.$$

Thus, for such a quaternion q, the imaginary part q' can be considered as a point (b, c, d) on a sphere of radius $\Phi|a|$:

$$b^2 + c^2 + d^2 = \Phi^2 a^2, \quad \text{or} \quad \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2 + \left(\frac{d}{a}\right)^2 = \Phi^2.$$

The number $\Phi^2 = \Phi + 1 = 2.6180339887498948\ldots$. The point $(b/a, c/a, d/a)$ is on the sphere of the radius Φ. If the quaternion q is a complex number, i.e., $q' = ib$, then this condition can be written as $b = \pm a\Phi$. Thus,

$$q = a + ia\Phi = a(1+i)\Phi, \quad \text{or} \quad q = a - ia\Phi = a(1-i)\Phi.$$

If $|a| > |q'|$ and these two parts are in the golden ratio, i.e., $|a|/|q'| = \Phi$,

$$|q| = \sqrt{a^2 + |q'|^2} = \sqrt{(1+\varphi^2)a^2} = |a|\sqrt{1+\varphi^2} = |a|\sqrt{2+\varphi}.$$

Here, we consider $1/\Phi = -\varphi$. Thus, the imaginary part q' of the quaternion q can be considered as point (b, c, d) on a sphere of radius $|a\varphi|$:

$$b^2 + c^2 + d^2 = a^2\varphi^2, \quad \text{or} \quad \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2 + \left(\frac{d}{a}\right)^2 = \varphi^2,$$

where $\varphi^2 = \varphi + 1 = 0.3819660112501052\ldots$.

Now, we generalize the results of the above two examples. When considering the polynomial equation

$$q^2 + 2\lambda q + v = 0 \tag{1.19}$$

with real coefficients λ and v, we can separate the two cases when the discriminant of the equation $\Delta = (\lambda^2 - v)$ is negative and non-negative.

Case $\Delta < 0$: The equation

$$z^2 + 2\lambda z + v = 0$$

has two solutions,

$$z_{1,2} = -\lambda \pm \sqrt{\lambda^2 - v} = -\lambda \pm i\sqrt{v - \lambda^2},$$

where i is the complex imaginary unit. The transformation of this equation into the space of quaternion numbers enlarges the set of solutions. Indeed, Eq. (1.19) can be written as

$$q^2 + 2\lambda q + v = (a^2 - |q'|^2 + 2aq') + 2\lambda(a + q') + v = 0$$

and reduced to the system of equations

$$\begin{cases} a^2 - |q'|^2 + 2a\lambda + v = 0 \\ (a + \lambda)q' = 0. \end{cases} \tag{1.20}$$

The $q' = 0$ case is not considered since the equation $a^2 + 2a\lambda + v = 0$ is not solvable in real numbers. Therefore, we consider the $a + \lambda = 0$ case, i.e., when $a = -\lambda$. From the first equation of Eq. (1.20), we obtain

$$|q'|^2 = a^2 + 2a\lambda + v = \lambda^2 - 2\lambda^2 + v = v - \lambda^2 = -\Delta > 0.$$

Thus, the solutions of Eq. (1.19) are all quaternion numbers of type

$$q = -\lambda + q', \quad \text{such that} \quad |q'| = \sqrt{-\Delta}.$$

The imaginary parts of these solutions are on a 3-D sphere of radius $r = \sqrt{-\Delta}$. In Example 7, which concerns the equation $q^2 - q + 2 = 0$, the numbers $\lambda = -1/2$ and $v = 2$. Therefore, $-\Delta = v - \lambda^2 = 2 - 1/4 = 7/4$, and $r = \sqrt{7}/2$.

Case $\Delta \geq 0$: The equation

$$x^2 + 2\lambda x + v = 0$$

has the real solutions

$$x_{1,2} = -\lambda \pm \sqrt{\lambda^2 - v} = -\lambda \pm \sqrt{\Delta}.$$

These two solutions relate to the $q' = 0$ case in Eq. (1.20). The condition $q' \neq 0$, or $a + \lambda = 0$, leads to the equality $|q'|^2 = v - \lambda^2 = -\Delta \leq 0$, which may only be satisfied for $q' = 0$. Thus, as in Example 9, quaternion arithmetic does not give a new solution of the quadratic equation that is solvable in real numbers.

P3. The quaternion conjugate of $q = a + q' = a + (bi + cj + dk)$ is the quaternion

$$\bar{q} = a - (bi + cj + dk),$$

such that

$$q\bar{q} = [a + (bi + cj + dk)][a - (bi + cj + dk)] = a^2 + b^2 + c^2 + d^2 \geq 0.$$

The operation of complex conjugate is linear, i.e.,

$$\overline{(q_1 + q_2)} = \bar{q}_1 + \bar{q}_2.$$

Indeed, the following calculations hold: $q_1 + q_2 = (a_1 + q'_1) + (a_2 + q'_2) = (a_1 + a_2) + (q'_1 + q'_2)$; therefore,

$$\overline{q_1 + q_2} = (a_1 + a_2) - (q'_1 + q'_2) = (a_1 - q'_1) + (a_2 - q'_2) = \bar{q}_1 + \bar{q}_2.$$

For any real number λ,

$$\overline{\lambda q} = \lambda \bar{q}.$$

It follows from Eq. (1.9) that the quaternion conjugate of multiplication $q_1 q_2$ equals

$$\overline{q_1 q_2} = -[a_1 q'_2 + a_2 q'_1] + a_1 a_2 - (b_1 b_2 + c_1 c_2 + d_1 d_2) - \begin{vmatrix} i & j & k \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix} \quad (1.21)$$

and

$$\bar{q}_2 \bar{q}_1 = -[a_1 q'_2 + a_2 q'_1] + a_1 a_2 - (b_1 b_2 + c_1 c_2 + d_1 d_2) + \begin{vmatrix} i & j & k \\ b_2 & c_2 & d_2 \\ b_1 & c_1 & d_1 \end{vmatrix}.$$

Thus, $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$ and might not equal $\bar{q}_1 \bar{q}_2$.

Example 1.13

Consider two quaternion numbers $q_1 = 1 + i - 2j + k$ and $q_2 = 1 + 2i - j - k$. The following holds for the conjugate of multiplication:

$$\overline{q_1 q_2} = \overline{(1 + i - 2j + k)(1 + 2i - j - k)} = \overline{-2 + 6i + 3k} = -2 - 6i - 3k$$

and $\bar{q}_2\bar{q}_1 = -2 - 6i - 3k$. The multiplication of their conjugate numbers $\bar{q}_1 = 1 - i + 2j - k$ and $\bar{q}_2 = 1 - 2i + j + k$ equals

$$\bar{q}_1\bar{q}_2 = (1 - i + 2j - k)(1 - 2i + j + k) = -2 + 6j + 3k.$$

P4. The module of the quaternion q equals

$$|q| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

In the particular case, when $q = q'$, the module $|q'| = \sqrt{b^2 + c^2 + d^2}$. In general, $|q| = \sqrt{a^2 + |q'|^2}$ and $|q| \geq |q'|$.

It is not difficult to see that the following holds for the operation of the quaternion conjugate:

$$\bar{\bar{q}} = q,$$

and

$$q + \bar{q} = 2a, \quad q - \bar{q} = 2q' = 2(bi + cj + dk).$$

The following calculations are valid:

$$|q_1q_2|^2 = (q_1q_2)\overline{q_1q_2} = (q_1q_2)(\bar{q}_2\bar{q}_1) = q_1|q_2|^2\bar{q}_1 = |q_2|^2(q_1\bar{q}_1) = |q_2|^2|q_1|^2.$$

Thus,

$$|q_1q_2| = |q_2||q_1| = |q_1||q_2|.$$

It is not difficult to show the property of triangle inequality,

$$|q_1 + q_2| \leq |q_1| + |q_2|.$$

To do so, it is enough to prove that for any quaternion $q = a + (ib + cd + kd)$, the inequality $|q + 1| \leq |q| + 1$ holds. The following calculations hold:

$$\begin{aligned} |q + 1|^2 &= (q + 1)(\bar{q} + 1) = |q|^2 + 1 + (q + \bar{q}) = |q|^2 + 1 + 2a \\ &\leq |q|^2 + 1 + 2|a| \leq |q|^2 + 1 + 2|q| = (|q| + 1)^2. \end{aligned}$$

P5. The inverse q^{-1} to the quaternion $q \neq 0$ is defined as the quaternion number such that $q^{-1}q = 1$. This is the inverse number when multiplying q from the left and is defined (from the definition $\bar{q}q = |q|^2$) as the multiplication of the quaternion q by the real number

$$q^{-1} = \frac{1}{|q|^2}\bar{q}.$$

The inverse from the left and right are the same, i.e., $qq^{-1} = q^{-1}q = 1$. Note that if q is a unit quaternion, i.e., $|q| = 1$, then $q^{-1} = \bar{q}$.

Example 1.14

Consider the quaternion number $q = 1 + 2i - 3j + 4k$. Then, $\bar{q} = 1 - 2i + 3j - 4k$, and the inverse q^{-1} is calculated as

$$(1+2i-3j+4k)^{-1} = \frac{1}{1+2^2+3^2+4^2}(1-2i+3j-4k) = \frac{1}{30}(1-2i+3j-4k).$$

The following property holds for the inverse of multiplication:

$$(q_1q_2)^{-1} = \frac{1}{|q_1q_2|^2}\overline{q_1q_2} = \frac{1}{|q_1|^2|q_2|^2}\bar{q}_2\bar{q}_1 = \frac{\bar{q}_2}{|q_2|^2}\frac{\bar{q}_1}{|q_1|^2} = (q_2)^{-1}(q_1)^{-1}.$$

In the particular case when $q_1 = q_2$, we obtain $(q_1^2)^{-1} = (q_1^{-1})^2$.

P6. Multiplication is not a commutative operation; therefore, the operation of division of one quaternion q_1 by another $q_2 \neq 0$ is considered when solving one of the following two equations:

$$q_1 = q_2q \quad \text{and} \quad q_1 = qq_2.$$

For division from the left, multiplication of the equality $q_1 = q_2q$ by $\bar{q}_2$ from the left results in

$$\bar{q}_2q_2q = \bar{q}_2q_1 \quad \text{or} \quad |q_2|^2q = \bar{q}_2q_1.$$

Therefore, we obtain the solution

$$q = \frac{1}{|q_2|^2}\bar{q}_2q_1.$$

We can denote this division operation from the left as

$$q = q_l = \frac{1}{q_2}q_1.$$

Similarly, after multiplying both sides of equation $q_1 = qq_2$ by $\bar{q}_2$ from the right, we obtain the solution of this equation, which equals

$$q = \frac{1}{|q_2|^2}q_1\bar{q}_2 = q_1\frac{1}{|q_2|^2}\bar{q}_2.$$

We can denote this division from the right by

$$q = q_r = q_1\frac{1}{q_2}.$$

The divisions from the right and left are the same if $\bar{q}_2 q_1 = q_1 \bar{q}_2$, which can also be written as $\bar{q}_1 q_2 = q_2 \bar{q}_1$.

Example 1.15

Consider two quaternions $q_1 = 1 + i + k$ and $q_2 = 1 + 2i - j + k$. The left and right divisions of q_1 by q_2 are calculated as

$$q_l = \frac{1}{|q_2|^2} \bar{q}_2 q_1 = \frac{1}{1+4+1+1}(1 - 2i + j - k)(1 + i + k) = \frac{1}{7}(4 + 2j - k)$$

and

$$q_r = \frac{1}{|q_2|^2} q_1 \bar{q}_2 = \frac{1}{1+4+1+1}(1 + i + k)(1 - 2i + j - k) = \frac{1}{7}(4 - 2i + k).$$

We can see that these two numbers are different.

Below is the script of the MATLAB-based code "test_example10.m" to calculate the division of two quaternion numbers for this example.

```
%-----------------------------------------------
% test_example10.m
    q1=[1,1,0,1];                      % q1=1+i+0j+k
    q2=[1,2,-1,1];                     % q2=1+2i-j+k
    q2c=[q2(1),-q2(2:4)];              % conjugate to q2
    q2m=q2*q2';                        % |q2|^2=7
%-------------- Left Division (ql) -------------
    q2c1= mult2qs_direct(q2c,q1);  % 4 0 2 -1
    ql=(1/q2m)*q2c1;          % 0.5714 0 0.2857 -0.1429
%   checking the result
    mult2qs_direct(q2,ql);            % 1 1 0 1 (q1)
%-------------- Right Division (qr) -------------
    q12c=mult2qs_direct(q1,q2c); % 4 -2 0 1
    qr=(1/q2m)*q12c;    % 0.5714 -0.2857 0 0.1429
%   checking the result
    mult2qs_direct(qr,q2);            % 1 1 0 1 (q1)
%-----------------------------------------------
```

P7. It directly follows from Eq. (1.9) that if the numbers q_1 and q_2 are pure quaternions, i.e., $a_1 = a_2 = 0$, then

$$q = q_1 q_2 = -(b_1 b_2 + c_1 c_2 + d_1 d_2) + \begin{vmatrix} i & j & k \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix}. \tag{1.22}$$

When $q_1 = q_2$, we obtain from this Eq. (1.22) the following:

$$(q_1)^2 = -(b_1^2 + c_1^2 + d_1^2) + \begin{vmatrix} i & j & k \\ b_1 & c_1 & d_1 \\ b_1 & c_1 & d_1 \end{vmatrix} = -(b_1^2 + c_1^2 + d_1^2) = -|q_1|^2. \quad (1.23)$$

Therefore, for the pure unit quaternion q_1, i.e., $|q_1| = 1$, its square $q_1^2 = -1$.

P8. For any two pure quaternion numbers q_1 and q_2, the following holds:

$$q_1q_2 - q_2q_1 = 2\begin{vmatrix} i & j & k \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix},$$

and this difference is zero (or $q_1q_2 = q_2q_1$) only if q_1 and q_2 are similar, i.e., $q_2 = Aq_2$, where A is a real number.

P9. We also use the notation $q = q_e + iq_i + jq_j + kq_k$ for the quaternion number $q = a + bi + cj + dk$, i.e., $q_e = a$, $q_i = b$, $q_j = c$, and $q_k = d$. The quaternion q can be considered as a vector

$$\mathbf{q} = (q_e, q_i, q_j, q_k)^T = (a, b, c, d)^T$$

in four-dimensional real space R^4, when the basic vectors of this space are $\mathbf{e} = (1, 0, 0, 0)^T$, $\mathbf{i} = (0, 1, 0, 0)^T$, $\mathbf{j} = (0, 0, 1, 0)^T$, and $\mathbf{k} = (0, 0, 0, 1)^T$. Here, the operation $(\cdot)^T$ stands for the transposition.

The direct calculation of the multiplication q_1q_2 results in the following:

$$\begin{aligned} q_1q_2 &= (a_1 + ib_1 + jc_1 + kd_1)(a_2 + ib_2 + jc_2 + kd_2) \\ &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + i(a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1) \\ &\quad + j(a_1c_2 + a_2c_1 - b_1d_2 + b_2d_1) + k(a_1d_2 + a_2d_1 + b_1c_2 - b_2c_1). \end{aligned}$$

This formula can also be obtained in symbolic programming in MATLAB. As an example, a simple script of the code "test_example.m" for symbolic calculation of the multiplication of two quaternions is given below.

```
% mult2q_symbolic.m    / Art Grigoryan, 2015
% the formula for multiplication q1q2 of two quaternions q1
  and q2
% -----------------------------------------------------
  syms a1 b1 c1 d1 as real;
  syms a2 b2 c2 d2 as real;
  g1 = [a1 b1 c1 d1];
  g2 = [a2 b2 c2 d2];
  q1q2 = mult2qs(g1,g2)
% -----------------------------------------------------
```

The script of the code "mult2qs.m" is given in the next script.

Therefore, in matrix form, the multiplication $q = q_1 q_2$ can be written as

$$\mathbf{q} = \mathbf{A}\mathbf{q}_2 = \begin{pmatrix} a_1 & -b_1 & -c_1 & -d_1 \\ b_1 & a_1 & -d_1 & c_1 \\ c_1 & d_1 & a_1 & -b_1 \\ d_1 & -c_1 & b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}, \tag{1.24}$$

where the four-dimensional (4-D) vectors are

$$\mathbf{q} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}.$$

The quaternion numbers $\mu = \mu_e + i\mu_i + j\mu_j + k\mu_k$ such that $\mu^2 = -1$ are pure quaternions. For instance, the numbers

$$\mu = \frac{i+j+k}{\sqrt{3}}, \quad \mu = \frac{i-j+k}{\sqrt{3}}, \quad \text{and} \quad \mu = \frac{i+j}{\sqrt{2}}$$

are pure unit quaternions. For such quaternions, $\mu_e = 0$ and the imaginary part lies on the unit sphere. It should be noted that for any real triplet $(\mu_i, \mu_j, \mu_k) \neq (0, 0, 0)$, the corresponding number

$$\mu = \frac{i\mu_i + j\mu_j + k\mu_k}{\sqrt{\mu_i^2 + \mu_j^2 + \mu_k^2}}$$

is a pure unit quaternion.

The equation $\mu^2 = -1$ can also be considered in matrix form as

$$\begin{pmatrix} \mu_e & -\mu_i & -\mu_j & -\mu_k \\ \mu_i & \mu_e & -\mu_k & \mu_j \\ \mu_j & \mu_k & \mu_e & -\mu_i \\ \mu_k & -\mu_j & \mu_i & \mu_e \end{pmatrix} \begin{pmatrix} \mu_e \\ \mu_i \\ \mu_j \\ \mu_k \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that

$$\mu_e^2 - \mu_i^2 - \mu_j^2 - \mu_k^2 = -1,$$
$$2\mu_e\mu_i = 0, \quad 2\mu_e\mu_j = 0, \quad 2\mu_e\mu_k = 0.$$

Therefore, $\mu_e = 0$ and $|\mu|^2 = \mu_i^2 + \mu_j^2 + \mu_k^2 = 1$, i.e., the point (μ_i, μ_j, μ_k) is on the unit sphere.

Below is the script of the MATLAB-based code-function "mult2qs.m" for calculating the multiplication of two quaternions by using the matrix **A**.

```
%----------------------------------------------------------
% call: muit2qs.m   / Art Grigoryan, UTSA January 10, 2015
% the matrix multiplication of 4-D quaternion-vectors
% q1=[a1,b1,c1,d1] * q2=[a2,b2,c2,d2]
function q12=mult2qs(q1,q2)
    a1=q1(1);  b1=q1(2);  c1=q1(3);  d1=q1(4);
    if size(q2,1)==1 q2=q2'; end  % q2 to be a column-vector
    A=[ a1 -b1 -c1 -d1
        b1  a1 -d1  c1
        c1  d1  a1 -b1
        d1 -c1  b1  a1];          % See Eq. (1.24)
    q12=A*q2;
%----------------------------------------------------------
```

This function as the function "mult2qs_direct.m" uses 16 real multiplications and 12 real additions in the multiplication of two quaternion numbers.

Matrix **A** in Eq. (1.24) is defined by the quaternion q_1, so we can write $\mathbf{A} = \mathbf{A}_{L;q_1}$. The subscript L stands for the vector q_1 on the left of q_1q_2. It is not difficult to see that the same multiplication $q = q_1q_2$ can be written by a matrix defined by the quaternion q_2 as

$$\mathbf{q} = \mathbf{A}_{R;q_2}\mathbf{q}_1 = \begin{pmatrix} a_2 & -b_2 & -c_2 & -d_2 \\ b_2 & a_2 & d_2 & -c_2 \\ c_2 & -d_2 & a_2 & b_2 \\ d_2 & c_2 & -b_2 & a_2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}, \tag{1.25}$$

where the 4-D vector $\mathbf{q}_1 = (a_1, b_1, c_1, d_1)^T$. Here, we use the subscript R for multiplication by the matrix that is generated by the quaternion from the right in the multiplication q_1q_2.

Below is the code-function "mult2qsR.m" for calculating the multiplication of two quaternions by using the matrix $\mathbf{A}_{R;q_2}$.

```
%----------------------------------------------------------
% call: muit2qsR.m  / Art Grigoryan, UTSA January 10, 2015
% the matrix multiplication of 4-D quaternion-vectors
% q1=[a1,b1,c1,d1] * q2=[a2,b2,c2,d2]
function q12=mult2qsR(q1,q2)
    a2=q2(1);  b2=q2(2);  c2=q2(3);  d2=q2(4);
    if size(q1,1)==1 q1=q1'; end  % q1 to be a column-vector
```

```
    Aq2 = [ a2 -b2 -c2 -d2
            b2  a2  d2 -c2
            c2 -d2  a2  b2
            d2  c2 -b2  a2];      % See Eq. (1.25)
    q12 = Aq2*q1;
%-----------------------------------------------------------
```

We now describe an important property of the inverse matrices to matrices **A**, but first we use a simple script of the calculation of such matrices.

Example 1.16

Consider the quaternion number $q = 1 + 2i + 3j - 4k$ with $|q|^2 = 15$. If we run the code "inver2Amatrices.m," the script of which is given below, we obtain the following multiplication of the matrices with their transposition:

$$\mathbf{A}_{L;q} = \begin{pmatrix} 1 & -2 & -3 & 1 \\ 2 & 1 & 1 & 3 \\ 3 & -1 & 1 & -2 \\ -1 & -3 & 2 & 1 \end{pmatrix}, \quad \mathbf{A}^T_{L;q} = \begin{pmatrix} 1 & 2 & 3 & -1 \\ -2 & 1 & -1 & -3 \\ -3 & 1 & 1 & 2 \\ 1 & 3 & -2 & 1 \end{pmatrix}, \quad \mathbf{A}_{L;q} \cdot \mathbf{A}^T_{L;q} = 15\mathbf{I}_4,$$

where $\mathbf{I}_4$ denotes the 4×4 identity matrix.

```
%-----------------------------------------------------------
% call: inver2Amatrices.m / Artyom Grigoryan, January 2, 2016
% calculate inverse matrices of the left and right
  multiplication
    q = [1,2,3, - 1];
    % q = q/norm(q);  % |q|^2 = norm(q)^2 = 15
    a = q(1);  b = q(2);  c = q(3);  d = q(4);
    A1 = [ a -b -c -d
           b  a -d  c
           c  d  a -b
           d -c  b  a];      % See Eq. (1.17)
    A2 = [ a -b -c -d
           b  a  d -c
           c -d  a  b
           d  c - b  a];     % See Eq. (1.18)
d1 = det(A1); d2 = det(A2);  % d1 = d2 = 225 which is 15*15
what_weget = A1'*A1;
what_weget = A2'*A2;
%  15   0   0   0
%   0  15   0   0
%   0   0  15   0
%   0   0   0  15
% if open the command q=q/norm(q), then we get the identity
  matrix
%-----------------------------------------------------------
```

Similarly, for multiplication from the right, we have

$$\mathbf{A}_{R;q}=\begin{pmatrix}1&-2&-3&1\\2&1&-1&-3\\3&1&1&2\\-1&3&-2&1\end{pmatrix},\quad \mathbf{A}_{R;q}^{T}=\begin{pmatrix}1&2&3&-1\\-2&1&1&3\\-3&-1&1&-2\\1&-3&2&1\end{pmatrix},\quad \mathbf{A}_{R;q}\cdot\mathbf{A}_{R;q}^{T}=15\mathbf{I}_4.$$

We note that $|q|^2=15$ and have the following inverse matrices:

$$\mathbf{A}_{L;q}^{-1}=\frac{1}{15}\mathbf{A}_{L;q}^{T}=\frac{1}{15}\begin{pmatrix}1&2&3&-1\\-2&1&-1&-3\\-3&1&1&2\\1&3&-2&1\end{pmatrix}$$

and

$$\mathbf{A}_{R;q}^{-1}=\frac{1}{15}\mathbf{A}_{R;q}^{T}=\frac{1}{15}\begin{pmatrix}1&2&3&-1\\-2&1&1&3\\-3&-1&1&-2\\1&-3&2&1\end{pmatrix}.$$

Also, we can see that the determinant of both matrices is $225=(15)^2$.

In general, we have the following for matrices $\mathbf{A}$:

$$\det(\mathbf{A}_{L;q})=\det(\mathbf{A}_{R;q})=|q|^4 \tag{1.26}$$

and

$$\mathbf{A}_{L;q}^{-1}=\frac{1}{|q|^2}\mathbf{A}_{L;q}^{T}\quad\text{and}\quad \mathbf{A}_{R;q}^{-1}=\frac{1}{|q|^2}\mathbf{A}_{R;q}^{T}. \tag{1.27}$$

If the quaternion is a unit quaternion, i.e., $|q|=1$, these inverse matrices are transpose matrices.

1.2 Vector Space and Pure Quaternions

To describe the geometry of the multiplication presented in the previous section, we consider the Cartesian coordinate system in 3-D space, vectors $\mathbf{q}'=[b,c,d]$ with original $(0,0,0)$, and the terminal points (b,c,d). Along three perpendicular X,- Y,- and Z-axes in this system, we can measure the unit vectors as

$$\mathbf{i}=[1,0,0],\qquad \mathbf{j}=[0,1,0],\qquad \text{and}\qquad \mathbf{k}=[0,0,1],$$

respectively, as shown in Fig. 1.10. The space of quaternion numbers is 4-D, and the Cartesian model in Fig. 1.10 is shown only to present the space of the

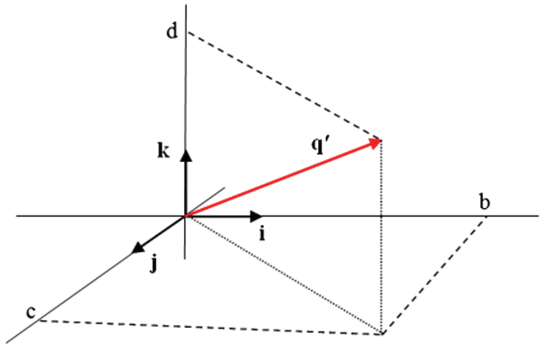

Figure 1.10 The Cartesian coordinate system in the 3-D space.

imaginary components of the quaternion numbers. In general, it is not necessary to draw the i-, j-, and k-axes perpendicular to each other in the space of quaternions.

A pure quaternion number $q' = (b, c, d)$ can be presented by the vector $\mathbf{q}' = [b, c, d]$ in this vector space. The sum of pure quaternion numbers and multiplication by real numbers correspond to similar operations on vectors. Thus, we can associate vectors with pure quaternion numbers and call them *vector quaternions*. Given a quaternion $q = a + q'$, a is the real part of q, and q' is *the vector part* of q.

In vector space, two interesting operations, the inner product and vector product, are described. These operations are used in many applications of vector algebra and mechanics, when calculating the work done by a force, or finding the normal vector to the plane, moment of a force, velocity of a rotating body, etc. We now describe these operations performed on vector quaternions.

1.2.1 Inner product (or dot product)

The operation of the inner product of two vectors $\mathbf{q}'_1 = [b_1, c_1, d_1]$ and $\mathbf{q}'_2 = [b_2, c_2, d_2]$ is defined as

$$(\mathbf{q}'_1, \mathbf{q}'_2) = b_1 b_2 + c_1 c_2 + d_1 d_2. \tag{1.28}$$

Thus, the real part of the multiplication in Eq. (1.22) is the inner product with the sign minus.

In vector space, the inner product is defined by the following rule: for each two elements $\mathbf{q}'_1$ and $\mathbf{q}'_2$ of the space with which a real number is associated, *the inner product* that is denoted as $(\mathbf{q}'_1, \mathbf{q}'_2)$ has the following four properties (axioms):

1. $(\mathbf{q}'_1, \mathbf{q}'_2) = (\mathbf{q}'_2, \mathbf{q}'_1)$.
2. $(\mathbf{q}'_1 + \mathbf{q}'_3, \mathbf{q}'_2) = (\mathbf{q}'_1, \mathbf{q}'_2) + (\mathbf{q}'_3, \mathbf{q}'_2)$.
3. $(\lambda\mathbf{q}'_1, \mathbf{q}'_2) = \lambda(\mathbf{q}'_1, \mathbf{q}'_2)$, for any real number λ.
4. $(\mathbf{q}', \mathbf{q}') \geq 0$, and $(\mathbf{q}', \mathbf{q}') = 0$ if only $\mathbf{q}' = 0$.

The value of $(\mathbf{q}', \mathbf{q}')$ is denoted by $|\mathbf{q}'|^2$, and the positive number $|\mathbf{q}'| = \sqrt{(\mathbf{q}', \mathbf{q}')}$ is called *the length* of vector $\mathbf{q}'$. As in real Euclidean space, the concept of the angle between the vectors can be defined in the vector space of pure quaternions. Indeed, it directly follows from the above axioms, which for given vectors $\mathbf{q}_1'$ and $\mathbf{q}_2'$, the following polynomial is non-negative:

$$p(\lambda) = (\lambda\mathbf{q}_1' - \mathbf{q}_2', \lambda\mathbf{q}_1' - \mathbf{q}_2') = |\lambda\mathbf{q}_1' - \mathbf{q}_2'|^2 \geq 0.$$

Here, λ is a real variable. Thus,

$$\lambda^2(\mathbf{q}_1', \mathbf{q}_1') - 2\lambda(\mathbf{q}_1', \mathbf{q}_2') + (\mathbf{q}_2', \mathbf{q}_2') \geq 0,$$

and the discriminant of this equation

$$\Delta = (\mathbf{q}_1', \mathbf{q}_2')^2 - (\mathbf{q}_1', \mathbf{q}_1')(\mathbf{q}_2', \mathbf{q}_2') = (\mathbf{q}_1', \mathbf{q}_2')^2 - |\mathbf{q}_1'|^2|\mathbf{q}_2'|^2 \leq 0.$$

Therefore, $(\mathbf{q}_1', \mathbf{q}_2')^2/(|\mathbf{q}_1'|^2|\mathbf{q}_2'|^2) \leq 1$, or

$$-1 \leq \frac{(\mathbf{q}_1', \mathbf{q}_2')}{|\mathbf{q}_1'||\mathbf{q}_2'|} \leq 1.$$

The fraction in this expression can be considered as a cosine of angle ϑ:

$$\cos(\vartheta) = \frac{(\mathbf{q}_1', \mathbf{q}_2')}{|\mathbf{q}_1'||\mathbf{q}_2'|}, \quad \text{and} \quad \sin(\vartheta) = \sqrt{1 - \frac{(\mathbf{q}_1', \mathbf{q}_2')^2}{|\mathbf{q}_1'|^2|\mathbf{q}_2'|^2}}.$$

This angle is called the angle between vectors $\mathbf{q}_1'$ and $\mathbf{q}_2'$. Since

$$(\mathbf{q}_1', \mathbf{q}_2') = |\mathbf{q}_1'||\mathbf{q}_2'|\cos(\vartheta), \tag{1.29}$$

we can call two vectors *perpendicular* if $(\mathbf{q}_1', \mathbf{q}_2') = 0$. This condition can be denoted as $\mathbf{q}_1' \perp \mathbf{q}_2'$.

It directly follows from Eq. (1.22) that if vectors $\mathbf{q}_1'$ and $\mathbf{q}_2'$ are perpendicular, then the following holds for the pure quaternions $q_1 = q_1'$ and $q_2 = q_1'$:

$$q_1q_2 = \begin{vmatrix} i & j & k \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix} = -\begin{vmatrix} i & j & k \\ b_2 & c_2 & d_2 \\ b_1 & c_1 & d_1 \end{vmatrix} = -q_2q_1. \tag{1.30}$$

Also, if $q_1q_2 = -q_2q_1$, then q_1 and q_2 are pure quaternions, and the corresponding vectors $\mathbf{q}_1'$ and $\mathbf{q}_2'$ are perpendicular.

Since

$$q_1'q_2' = -(\mathbf{q}_1', \mathbf{q}_2') + \begin{vmatrix} i & j & k \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix},$$

the real part of $q_1'q_2'$ is zero if vectors $\mathbf{q}_1'$ and $\mathbf{q}_2'$ are perpendicular.

Example 1.17

Consider three vector quaternions $\mathbf{q}_1' = (1, 2 - 3)$, $\mathbf{q}_2' = (-1, 5, 3)$, and $\mathbf{q}_3' = (2, 1, 2)$. The first two vectors are orthogonal,

$$(\mathbf{q}_1', \mathbf{q}_2') = 1(-1) + 2(5) - 3(3) = 0,$$

and

$$q_1'q_2' = (1 + 2j - 3k)(-i + 5j + 3k) = 21i + 7k.$$

One can verify this result by using the calculation of the determinant:

$$\begin{vmatrix} i & j & k \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ -1 & 5 & 3 \end{vmatrix} = i(6 + 15) - j(3 - 3) + k(5 + 2) = 21i + 7k.$$

These two vectors are not perpendicular to vector $\mathbf{q}_3'$. Indeed,

$$(\mathbf{q}_1', \mathbf{q}_3') = 1(2) + 2(1) - 3(2) = -2,$$
$$(\mathbf{q}_2', \mathbf{q}_3') = -1(2) + 5(1) + 3(2) = 9.$$

The angle $\vartheta_{1,3}$ between vectors $\mathbf{q}_1'$ and $\mathbf{q}_3'$ is calculated by

$$\cos(\vartheta_{1,3}) = \frac{(\mathbf{q}_1', \mathbf{q}_3')}{|\mathbf{q}_1'||\mathbf{q}_3'|} = \frac{-2}{\sqrt{1+4+9}\sqrt{4+1+4}} = \frac{-2}{3\sqrt{14}} = -0.1782,$$

$$\vartheta_{1,3} = \arccos\left(-\frac{2}{3\sqrt{14}}\right) = 1.7499 \text{ rad, or } 100.2634°.$$

The angle $\vartheta_{2,3}$ between vectors $\mathbf{q}_2'$ and $\mathbf{q}_3'$ is calculated by

$$\cos(\vartheta_{2,3}) = \frac{(\mathbf{q}_2', \mathbf{q}_3')}{|\mathbf{q}_2'||\mathbf{q}_3'|} = \frac{9}{\sqrt{1+25+9}\sqrt{4+1+4}} = \frac{3}{\sqrt{35}} = 0.5071,$$

$$\vartheta_{2,3} = \arccos\left(\frac{3}{\sqrt{35}}\right) = 1.0390 \text{ rad, or } 59.5296°.$$

1.2.2 Vector product

In vector space, we can consider the symbolic determinate

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix} = \mathbf{i}(c_1d_2 - d_1c_2) - \mathbf{j}(b_1d_2 - d_1b_2) + \mathbf{k}(b_1c_2 - c_1b_2). \quad (1.31)$$

According to Eq. (1.22), the vector with the terminal point

$$[(c_1d_2 - d_1c_2), -(b_1d_2 - d_1b_2), (b_1c_2 - c_1b_2)]$$

describes the whole imaginary part of the multiplication $q = q_1q_2$, when $a_1 = a_2 = 0$. We also can say that this point corresponds to $(q_1'q_2')$. This vector is denoted by $\mathbf{q}_1' \times \mathbf{q}_2'$ or $[\mathbf{q}_1', \mathbf{q}_2']$ and is called *the vector product* of vectors $\mathbf{q}_1'$ and $\mathbf{q}_2'$.

It is not difficult to see that

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_2 & c_2 & d_2 \\ b_1 & c_1 & d_1 \end{vmatrix} = \mathbf{i}(c_2d_1 - d_2c_1) - \mathbf{j}(b_2d_1 - d_2b_1) + \mathbf{k}(b_2c_1 - c_2b_1) = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix},$$

and, therefore,

$$\mathbf{q}_2' \times \mathbf{q}_1' = -\mathbf{q}_1' \times \mathbf{q}_2'.$$

The vector product is perpendicular to vectors $\mathbf{q}_1'$ and $\mathbf{q}_2'$. To verify this property, it is sufficient to show that the real parts of vectors $\mathbf{q}_1'(\mathbf{q}_1' \times \mathbf{q}_2')$ and $\mathbf{q}_2'(\mathbf{q}_1' \times \mathbf{q}_2')$ are zero. The real part of the first multiplication can be calculated as

$$b_1(c_1d_2 - d_1c_2) - c_1(b_1d_2 - d_1b_2) + d_1(b_1c_2 - c_1b_2) = 0.$$

The real part of the second multiplication is similarly calculated and equals zero, too. The three perpendicular vectors $\mathbf{q}_1' \times \mathbf{q}_2'$, $\mathbf{q}_1'$, and $\mathbf{q}_2'$ are oriented in space similarly to the way unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are oriented. The orientation of these unit vectors can be considered in the chosen right- or left-handed Cartesian coordinate system in 3-D space.

The length of the vector product can be calculated by

$$\begin{aligned}
|\mathbf{q}_1' \times \mathbf{q}_2'|^2 &= (c_1d_2 - d_1c_2)^2 + (b_1d_2 - d_1b_2)^2 + (b_1c_2 - c_1b_2)^2 \\
&= (c_1^2d_2^2 - 2c_1c_2d_1d_2 + d_1^2c_2^2) + (b_1^2d_2^2 - 2b_1b_2d_1d_2 + d_1^2b_2^2) \\
&\quad + (b_1^2c_2^2 - 2b_1b_2c_1c_2 + c_1^2b_2^2) \\
&= (b_1^2 + c_1^2 + d_1^2)(b_2^2 + c_2^2 + d_2^2) - c_1^2c_2^2 - b_1^2b_2^2 - d_1^2d_2^2 \\
&\quad - 2c_1c_2d_1d_2 - 2b_1b_2d_1d_2 - 2b_1b_2c_1c_2 \\
&= (b_1^2 + c_1^2 + d_1^2)(b_2^2 + c_2^2 + d_2^2) - (b_1b_2 + c_1c_2 + d_1d_2)^2.
\end{aligned}$$

Because

$$\begin{aligned}
|\mathbf{q}_1'|^2 &= b_1^2 + c_1^2 + d_1^2,\\
|\mathbf{q}_2'|^2 &= b_2^2 + c_2^2 + d_2^2,\\
(\mathbf{q}_1', \mathbf{q}_2') &= |\mathbf{q}_1'||\mathbf{q}_2'|\cos(\vartheta) = b_1 b_2 + c_1 c_2 + d_1 d_2,
\end{aligned} \tag{1.32}$$

where ϑ is the angle between the vector quaternions, we obtain

$$|\mathbf{q}_1' \times \mathbf{q}_2'|^2 = |\mathbf{q}_1'|^2|\mathbf{q}_2'|^2 - |\mathbf{q}_1'|^2|\mathbf{q}_2'|^2\cos^2(\vartheta) = |\mathbf{q}_1'|^2|\mathbf{q}_2'|^2\sin^2(\vartheta).$$

Thus, the length of the vector products equals

$$|\mathbf{q}_1' \times \mathbf{q}_2'| = |\mathbf{q}_1'||\mathbf{q}_2'|\sin(\vartheta), \tag{1.33}$$

when considering $\sin(\vartheta)$ to be non-negative. It is the area of the parallelogram composed on vectors $\mathbf{q}_1'$ and $\mathbf{q}_2'$. If the vectors are perpendicular, then $\vartheta = \pi/2$, $\sin(\vartheta) = 1$, and

$$|\mathbf{q}_1' \times \mathbf{q}_2'| = |\mathbf{q}_1'||\mathbf{q}_2'|. \tag{1.34}$$

Now, we can define the inner product and vector product of quaternions as

$$(q_1, q_2) = (\mathbf{q}_1', \mathbf{q}_2') \quad \text{and} \quad (q_1 \times q_2) = (\mathbf{q}_1' \times \mathbf{q}_2'),$$

and consider pure quaternions as *vector quaternions*. The multiplication of pure quaternion numbers unites the inner and vector products:

$$q = q_1 q_2 = -(q_1, q_2) + (q_1 \times q_2). \tag{1.35}$$

One can notice that

$$q_2 q_1 = -(q_2, q_1) + (q_2 \times q_1) = -(q_1, q_2) - (q_1 \times q_2) = q_1 q_2 - 2(q_1 \times q_2).$$

In the general case of quaternions, the multiplication of two quaternions in Eq. (1.9) can be written as

$$q = q_1 q_2 = a_1 a_2 - (q_1, q_2) + [a_1 q_2' + a_2 q_1'] + (q_1 \times q_2). \tag{1.36}$$

The real part of this vector is the minus inner product of the imaginary parts of the quaternions,

$$a = a_1 a_2 - (q_1, q_2),$$

and the imaginary part of the multiplication is the sum of vector quaternions,

$$q' = [a_1 q_2' + a_2 q_1'] + (q_1 \times q_2).$$

The first vector in the square brackets presents a vector in the "$(\mathbf{q}_1') - (\mathbf{q}_2')$" plane, and the second vector is the vector product of the imaginary parts of the quaternions, which is perpendicular to this plane.

Example 1.18

Consider two quaternions $q_1 = 1 + i - j + 2k$ and $q_2 = 1 + 2i + j - 3k$. The vector quaternions are $\mathbf{q}_1' = (1, -1, 2)$ and $\mathbf{q}_2' = (2, 1, -3)$. The inner product equals

$$(q_1', q_2') = 1 \cdot 2 + (-1) \cdot 1 + 2 \cdot (-3) = -5,$$

and $a = 1 - (-5) = 6$. The vector product is calculated as

$$(q_1' \times q_2') = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \mathbf{i}(3 - 2) - \mathbf{j}(-3 - 4) + \mathbf{k}(1 + 2) = 1\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$$

and can be written as the vector with the terminal point $(1, 7, 3)$. The imaginary part of $q_1 q_2$ is

$$\begin{aligned} q' &= [a_1 q_2' + a_2 q_1'] + (q_1' \times q_2') = [(2i + j - 3k) + (i - j + 2k)] + (i + 7j + 3k) \\ &= [3i - k] + (i + 7j + 3k) = 4i + 7j + 2k; \end{aligned}$$

therefore, $q = a + q' = 6 + 4i + 7j + 2k$.

1.3 Quaternion Multiplication and Rotation

Quaternion multiplication allows one to describe a rotation of one vector in 3-D space around any other vector. In this section, we show a few examples of rotations by multiplication. First, we consider the multiplication of two quaternions, one of which is a pure quaternion. The geometry of such multiplications with rotations in the 3-D subspace of vectors presenting quaternion vectors is described by using the matrix representation of multiplication as well as the polar form of quaternions.

1.3.1 Multiplication and sum of elementary rotations

Consider the quaternions in the space R^4 as vectors. When multiplying the quaternion $\mathbf{q}_1 = (a_1, b_1, c_1, d_1)^T$ by the vector quaternion $\mathbf{q}_2 = (0, b_2, c_2, d_2)^T$, the real component of the multiplication is $-(b_1 b_2 + c_1 c_2 + d_1 d_2)$. It follows from Eq. (1.24),

$$\mathbf{q} = \mathbf{A}\mathbf{q}_2 = \begin{pmatrix} a_1 & -b_1 & -c_1 & -d_1 \\ b_1 & a_1 & -d_1 & c_1 \\ c_1 & d_1 & a_1 & -b_1 \\ d_1 & -c_1 & b_1 & a_1 \end{pmatrix} \begin{pmatrix} 0 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}, \tag{1.37}$$

that the matrix of multiplication in the 4-D real space $\mathbf{A}$ can be reduced to a (3×3) matrix:

$$\mathbf{B} = \begin{pmatrix} a_1 & -d_1 & c_1 \\ d_1 & a_1 & -b_1 \\ -c_1 & b_1 & a_1 \end{pmatrix},$$

when calculating the vector component of the multiplication $q_1 q_2$,

$$\begin{pmatrix} b \\ c \\ d \end{pmatrix} = \mathbf{B} \begin{pmatrix} b_2 \\ c_2 \\ d_2 \end{pmatrix}.$$

The point (b_2, c_2, d_2) is on a circle of radius $r_{2,j} = \sqrt{b_2^2 + c_2^2 = d_2^2}$.

This matrix can be represented as

$$\mathbf{B} = \mathbf{B}\left(\frac{a_1}{2}, 0, -c_1\right) + \mathbf{B}\left(\frac{a_1}{2}, d_1, 0\right) + \mathbf{B}\left(0, \frac{a_1}{2}, b_1\right),$$

where the matrices

$$\mathbf{B}_j = \mathbf{B}\left(\frac{a_1}{2}, 0, -c_1\right) = \begin{pmatrix} \frac{a_1}{2} & 0 & c_1 \\ 0 & 0 & 0 \\ -c_1 & 0 & \frac{a_1}{2} \end{pmatrix} \sim \mathbf{B}'_j = \begin{pmatrix} \frac{a_1}{2} & c_1 \\ -c_1 & \frac{a_1}{2} \end{pmatrix}, \ \det(\mathbf{B}'_j) = \frac{a_1^2}{4} + c_1^2,$$

$$\mathbf{B}_k = \mathbf{B}\left(\frac{a_1}{2}, d_1, 0\right) = \begin{pmatrix} \frac{a_1}{2} & -d_1 & 0 \\ d_1 & \frac{a_1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \mathbf{B}'_k = \begin{pmatrix} \frac{a_1}{2} & -d_1 \\ d_1 & \frac{a_1}{2} \end{pmatrix}, \ \det(\mathbf{B}'_k) = \frac{a_1^2}{4} + d_1^2,$$

$$\mathbf{B}_i = \mathbf{B}\left(0, \frac{a_1}{2}, b_1\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{a_1}{2} & -b_1 \\ 0 & b_1 & \frac{a_1}{2} \end{pmatrix} \sim \mathbf{B}'_i = \begin{pmatrix} \frac{a_1}{2} & -b_1 \\ b_1 & \frac{a_1}{2} \end{pmatrix}, \ \det(\mathbf{B}'_i) = \frac{a_1^2}{4} + b_1^2.$$

Each of these three matrices describes the rotation in the corresponding 2-D plane. Matrix $\mathbf{B}_j$ describes the rotation of the projection $(b_2, 0, d_2)$ of the vector (b_2, c_2, d_2) in the $(i - k)$-plane around the j-axis by the angle

$$\phi_j = -\arctan\left(\frac{2c_1}{a_1}\right).$$

Following this rotation, the point moves along the radial line from its circle of radius $r_{2,j} = \sqrt{b_2^2 + d_2^2}$ to another circle of radius amplified by a factor of $\lambda_j = \sqrt{\det(\mathbf{B}_j')}$. A similar rotation for complex numbers was described in Example 1.6 with Fig. 1.5.

In the first stage, matrix $\mathbf{B}_k$ describes the rotation in the $(i-j)$-plane, and $\mathbf{B}_i$ describes the rotation in the $(j-k)$-plane of projections $(b_2, c_2, 0)$ and $(0, b_2, c_2)$ by angles

$$\phi_k = \arctan\left(\frac{2d_1}{a_1}\right) \quad \text{and} \quad \phi_i = \arctan\left(\frac{2b_1}{a_1}\right),$$

respectively. Then, the rotated points move to circles with radii amplified by the factors of $\lambda_k = \sqrt{\det(\mathbf{B}_k')}$ and $\lambda_i = \sqrt{\det(\mathbf{B}_i')}$, respectively. Thus, matrix $\mathbf{B}$ is the sum of the three matrices of the elementary, or Givens rotations, as shown in Fig. 1.11. If q is the quaternion vector, i.e., $a_1 = 0$, all of these angles are considered to be $\pi/2$.

Statement 1.1 *Multiplication of the quaternion number $q_1 = a_1 + ib_1 + jc_1 + kd_1$ by a pure quaternion q_2 describes the rotation of the sum of three rotations of the vector quaternion q_2 around the i-, j-, and k-axes by the angles*

$$\phi_i = \arctan\left(\frac{2b_1}{a_1}\right), \quad \phi_j = \arctan\left(\frac{2c_1}{a_1}\right), \quad \textit{and} \quad \phi_k = \arctan\left(\frac{2d_1}{a_1}\right),$$

respectively, or all $\pi/2$ if $a_1 = 0$.

If q_2 is not a pure quaternion, $q_2 = a_2 + ib_2 + jc_2 + kd_2$ and $a_2 \neq 0$, we can write the multiplication $q_1 q_2$ in the following matrix form:

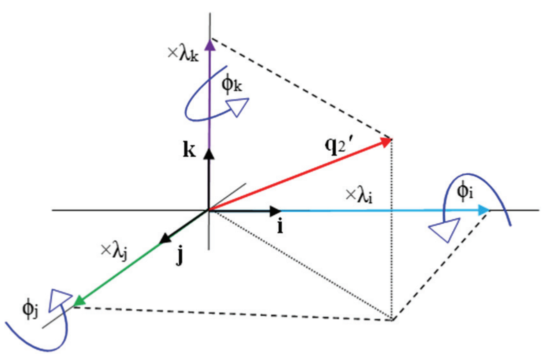

Figure 1.11 Three rotations around the axes in the vector subspace of quaternions.

$$\mathbf{A}\mathbf{q}_2 = \mathbf{A}\begin{pmatrix} a_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathbf{A}\begin{pmatrix} 0 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = a_2\mathbf{q}_1 + \begin{bmatrix} -(q_1', q_2') \\ \mathbf{B}\begin{pmatrix} b_2 \\ c_2 \\ d_2 \end{pmatrix} \end{bmatrix},$$

or we can write the multiplication as

$$\mathbf{A}\mathbf{q}_2 = \begin{bmatrix} a_2a_1 - (q_1', q_2') \\ a_2\mathbf{q}_1' + \mathbf{B}\mathbf{q}_2' \end{bmatrix}. \tag{1.38}$$

In the second line of matrix Eq. (1.38), the amplified vector $\mathbf{q}_1'$ is added to the combined rotation of vector $\mathbf{q}_2'$ by matrix $\mathbf{B}$. This is a geometrical interpretation of the quaternion multiplication in the general case.

Example 1.19

Consider two quaternions $q_1 = 2 + 1i + 3j + 4k$ and $q_2 = 0 + 2i + j - k$, where the numbers $a_1 = 2$, $b_1 = 1$, $c_1 = 3$, and $d_1 = 4$. The multiplication q_1q_2 of these two quaternions can be calculated as

$$\mathbf{q} = \mathbf{A}\mathbf{q}_2 = \begin{pmatrix} 2 & -1 & -3 & -4 \\ 1 & 2 & -4 & 3 \\ 3 & 4 & 2 & -1 \\ 4 & -3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 11 \\ -7 \end{pmatrix}.$$

Thus, the number $q = q_1q_2 = -1 - 3i + 11j - 7k$.

Figure 1.12 shows two vectors $\mathbf{q}_1' = (1, 3, 4)$ and $\mathbf{q}_2' = (2, 1, -1)$ and the vector $\mathbf{q}_{12}' = (-3, 11, -7)$. It can be shown that vector $\mathbf{q}_{12}'$ describes the rotation of vector $\mathbf{q}_2'$ around vector $\mathbf{q}_1'$ with a subsequent change in the length of vector $\mathbf{q}_2'$.

We now consider the rotations of the vector $\mathbf{q}_2' = (2, 1, -1)$ when calculating the imaginary part $q' = (-3i + 11j - 7k)$ of q. Matrix $\mathbf{B}$ is

$$\mathbf{B} = \begin{pmatrix} 2 & -4 & 3 \\ 4 & 2 & -1 \\ -3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Here, the matrix components and their determinants equal

$$\mathbf{B}_j = \mathbf{B}(1, 0, -3) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 1 \end{pmatrix} = \sqrt{10}\begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & 0 & 0 \\ -\frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{pmatrix}$$

and $\det(\mathbf{B}_j) = 1 + 9 = 10$;

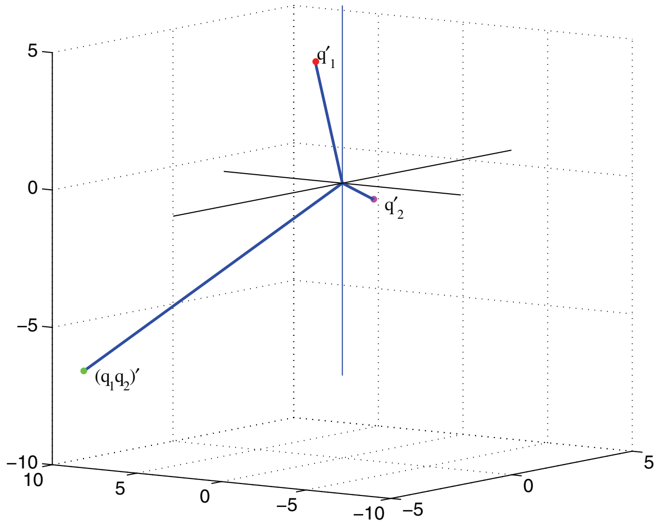

Figure 1.12 Vector $\mathbf{q}'_{12}$: vector $\mathbf{q}'_2$ rotation around vector $\mathbf{q}'_1$ with a subsequent change in length.

$$\mathbf{B}_k = \mathbf{B}(1,4,0) = \begin{pmatrix} 1 & -4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{17}\begin{pmatrix} \frac{1}{\sqrt{17}} & -\frac{4}{\sqrt{17}} & 0 \\ \frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\det(\mathbf{B}_k) = 1 + 16 = 17$; and

$$\mathbf{B}_i = \mathbf{B}(0,1,1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

and $\det(\mathbf{B}_i) = 1 + 1 = 2$.

Matrix $\mathbf{B}_j$ first rotates the projection vector $(2, 0, -1)$ along the circle of radius $r_{2,j} = \sqrt{5}$ around the j-axis by the angle

$$\phi_j = -\arctan\left(\frac{2c_1}{a_1}\right) = -\arctan(3) = 1.2490\,\text{rad},$$

or 71.5651° (see Fig. 1.13). The rotated projection is calculated as

$$\begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & 0 & 0 \\ -\frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 0 \\ -7 \end{pmatrix}.$$

Then, matrix $\mathbf{B}_j$ moves the rotated projection to the vector $(-1, 0, -7)$,

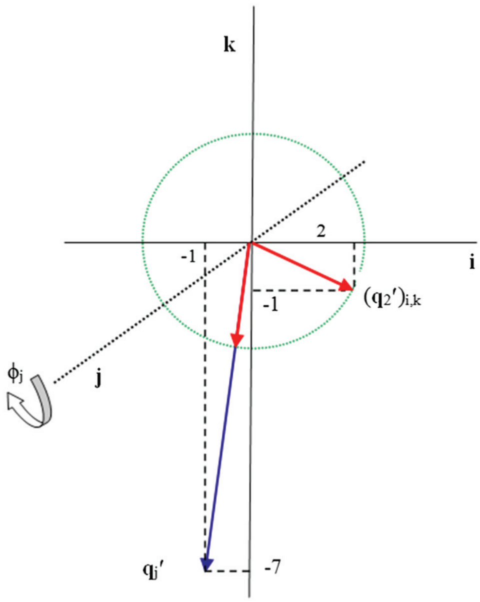

Figure 1.13 The rotation around the *j*-axis.

$$\mathbf{q}_j' = \sqrt{10}\cdot\begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & 0 & 0 \\ -\frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -7 \end{pmatrix}.$$

We can assume that the projection is first amplified by a factor of $\lambda_j = \sqrt{\det(\mathbf{B}_j)} = \sqrt{10}$, and then rotated by the angle ϕ_j,

$$\mathbf{q}_j' = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & 0 & 0 \\ -\frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{pmatrix}\left[\sqrt{10}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 0 \\ -7 \end{pmatrix},$$

along a circle of radius $r = \lambda_j r_{2,j} = \sqrt{50}$.

Matrix $\mathbf{B}_k$ rotates and moves the projection vector $(2, 1, 0)$ to the vector $(-2, 9, 0)$,

$$\mathbf{q}_k' = \begin{pmatrix} 1 & -4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \\ 0 \end{pmatrix},$$

around the k-axis by the angle

$$\phi_k = -\arctan\left(\frac{2d_1}{a_1}\right) = -\arctan(2) = 1.1071\,\text{rad},$$

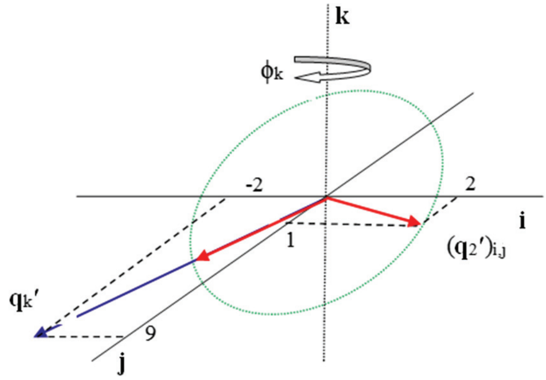

Figure 1.14 The rotation around the *k*-axis.

or 63.4349° (see Fig. 1.14). The rotated projection is calculated as

$$\begin{pmatrix} \frac{1}{\sqrt{17}} & -\frac{4}{\sqrt{17}} & 0 \\ \frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{17}} \begin{pmatrix} -2 \\ 9 \\ 0 \end{pmatrix}.$$

Then, matrix $\mathbf{B}_k$ moves the rotated projection to the vector $(-2, 9, 0)$ by multiplying the rotation by a factor of $\lambda_k = \sqrt{\det(\mathbf{B}_k)} = \sqrt{17}$.

Matrix $\mathbf{B}_i$ rotates and moves the vector $(0, 1, -1)$ to the vector $(-1, 0, -7)$. First, it rotates the vector as

$$\mathbf{q}_i' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

around the i-axis by the angle

$$\phi_i = -\arctan\left(\frac{2b_1}{a_1}\right) = -\arctan\left(\frac{1}{2}\right) = 0.4636\,\text{rad},$$

or 26.5651° (see Fig. 1.15). The rotated projection then is moved to the vector $(0, 2, 0)$, after multiplying by a factor of $\lambda_i = \sqrt{\det(\mathbf{B}_i)} = \sqrt{2}$.

We can verify the above calculations by

$$\mathbf{q}_j' + \mathbf{q}_k' + \mathbf{q}_i' = \begin{pmatrix} -1 \\ 0 \\ -7 \end{pmatrix} + \begin{pmatrix} -2 \\ 9 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 11 \\ -7 \end{pmatrix} = \mathbf{q}'.$$

The decomposition of the matrix $\mathbf{B}$ shows that vector $\mathbf{q}_{12}'$ can be obtained by a linear combination of three elementary rotations of the projections of the vector $\mathbf{q}_2'$ around the three axes of the Cartesian system.

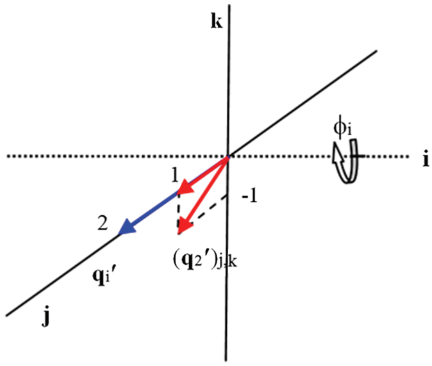

Figure 1.15 The rotation around the *i*-axis.

1.3.2 Rotation: Multiplication by a perpendicular vector

We consider the unit quaternion number (not pure) $q = a + q'$ and multiplication of this number by vector $\mathbf{s}'$, which is perpendicular to $\mathbf{q}'$.

Such a number can be represented as

$$q = \cos(\phi) + \frac{q'}{|q'|}\sin(\phi),$$

where the angle θ is defined from the condition $|q|^2 = a^2 + |q'|^2 = 1$ as

$$a = \cos(\phi) \quad \text{and} \quad |q'| = \sin(\phi).$$

The unit vector $\mathbf{p}' = \mathbf{q}'/|q'|$ is in the direction of the vector $\mathbf{q}'$.

Example 1.20

Consider the unit quaternion

$$q = \frac{1}{\sqrt{15}}(1 + i - 2j + 3k) = \frac{1}{\sqrt{15}} + \frac{i - 2j + 3k}{\sqrt{15}},$$

which can be written as

$$q = \frac{1}{\sqrt{15}} + \frac{i - 2j + 3k}{\sqrt{14}}\frac{\sqrt{14}}{15}.$$

In this case,

$$a = \frac{1}{\sqrt{15}}, \quad \mathbf{q}' = \frac{\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}}{\sqrt{15}}, \quad |q'| = \frac{\sqrt{14}}{15},$$

and the pure unit quaternion vector is

$$\mathbf{p}' = \frac{\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}}{\sqrt{14}} = 0.2673\mathbf{i} - 0.5345\mathbf{j} + 0.8018\mathbf{k}.$$

The angle θ is calculated from the equations

$$\cos(\phi) = \frac{1}{\sqrt{15}} = 0.2582 \quad \text{and} \quad \sin(\phi) = \frac{\sqrt{14}}{15} = 0.9661,$$

i.e., from the interval $[0, \pi/2]$ as

$$\phi = \arccos\left(\frac{1}{\sqrt{15}}\right) = 1.3096\,\text{rad},$$

or 75.0368°.

An illustration of the point (a, q') and vectors $\mathbf{p}'$ and $\mathbf{q}'$ are given in Fig. 1.16. The unit quaternion number q is presented as the point $(a, |q'|)$.

In the general case, when the module $|q|$ may be not 1, the quaternion q can be represented as

$$q = |q|[\cos(\phi) + p' \sin(\phi)], \quad \phi \in [0, \pi).$$

Thus, the number q is described as a triplet $(|q|, p', \phi)$, which is the polar form of the quaternion q. Below is the script of the MATLAB-based code “test_qinpolform.m” for Example 1.20, with the function “qinpolform.m” to calculate the polar form of q.

```
%-----------------------------------------------------
% test_qinpolform   / Art Grigoryan, November 10, 2015
    q= [1,1, -2,3]/sqrt(15);       % for Example 1.20
    [q_abs,p_vector,phi] = qinpolform(q);
```

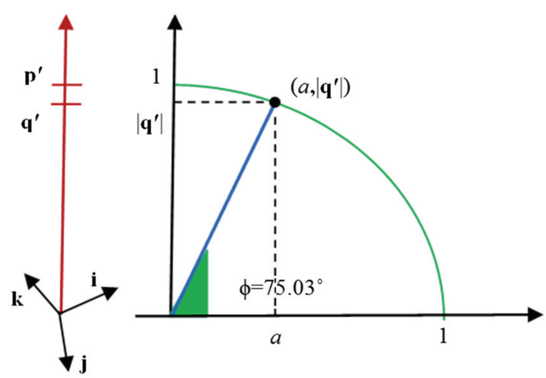

Figure 1.16 Components of the unit quaternion number $q = a + q'$

```
   % The result is: 1, (0.2673,−0.5345,0.8018), and
     1.3096 rad/s
%--------------------------------------------------
% call: qinpolform.m
% To calculate the polar form of a quaternion number
%      q=|q|[cos(phi) + p'sin(phi)]
%      as the triplet (|q|,p',phi).
  function [q_abs,p_vector,phi] = qinpolform(q)
    q_abs=norm(q);                % is |q|
    qn=q/q_abs;                   % q/|q|
    qim=qn(2:4);                  % imaginary part of qn
    qim_abs=norm(qim);            % is |qim|
    p_vector =qim/qim_abs;        % unit vector p'
    phi=acos(qn(1));              % angle phi
%--------------------------------------------------
```

The multiplication $q\mathbf{s}'$ equals

$$q\mathbf{s}' = [\cos(\phi) + \mathbf{p}' \sin(\phi)]\mathbf{s}' = \mathbf{s}' \cos(\phi) + (\mathbf{p}'\mathbf{s}') \sin(\phi).$$

Here, the multiplication of the vectors is calculated as

$$\mathbf{p}'\mathbf{s}' = -(\mathbf{p}', \mathbf{s}') + \mathbf{p}' \times \mathbf{s}' = \mathbf{p}' \times \mathbf{s}'.$$

We could also come to this result by writing the multiplication $q\mathbf{s}'$ as

$$\begin{aligned} q\mathbf{s}' &= a\mathbf{s}' - (\mathbf{q}', \mathbf{s}') + \mathbf{q}' \times \mathbf{s}' = a\mathbf{s}' + \mathbf{q}' \times \mathbf{s}' = a\mathbf{s}' + (|\mathbf{q}'|\mathbf{p}') \times \mathbf{s}' \\ &= a\mathbf{s}' + |\mathbf{q}'|(\mathbf{p}' \times \mathbf{s}') = \mathbf{s}' \cos(\phi) + (\mathbf{p}' \times \mathbf{s}') \sin(\phi). \end{aligned}$$

As shown above, the vector product $\mathbf{p}' \times \mathbf{s}'$ is a vector that is perpendicular to both vectors $\mathbf{p}'$ and $\mathbf{s}'$. According to Eq. (1.34), the length of the vector product equals

$$|\mathbf{p}' \times \mathbf{s}'| = |\mathbf{p}'||\mathbf{s}'| = |\mathbf{s}'|.$$

Denoting the vector product $\mathbf{p}' \times \mathbf{s}'$ by $\mathbf{v}'$, we can write $q\mathbf{s}' = \mathbf{s}' \cos(\phi) + \mathbf{v}' \sin(\phi)$. Both vectors $\mathbf{s}'$ and $\mathbf{v}'$ have the same length of $|\mathbf{s}'|$. Therefore, this equation describes the rotation of vector $\mathbf{s}'$ around vector $\mathbf{q}'$, or around $\mathbf{p}'$ by the angle θ as shown in Fig. 1.17. Vector $\mathbf{v}'$ is perpendicular to $\mathbf{s}'$, and we may say that $\mathbf{v}'$ is the result of the rotation of $\mathbf{s}'$ by 90° around the vector $\mathbf{p}'$. Thus, the following can be stated.

Statement 1.2 *The multiplication $q\mathbf{s}'$ describes the rotation of the perpendicular vector $\mathbf{s}'$ around the unit vector $\mathbf{p}'$ by the angle ϕ:*

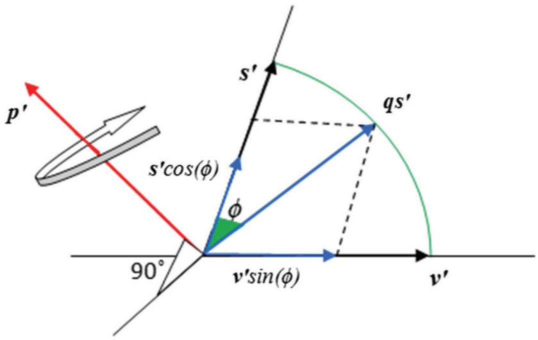

Figure 1.17 Rotation of vector $\mathbf{s}'$ around the unit vector $\mathbf{p}'$ by the angle ϕ.

$$qs' = \mathbf{s}'\cos(\phi) + v'\sin(\phi). \tag{1.39}$$

Example 1.21

Consider the unit quaternion number

$$q = a + q' = \frac{1}{\sqrt{15}} + \frac{1}{\sqrt{15}}(i - 2j + 3k) = \cos(\phi) + p'\sin(\phi).$$

As shown in Example 1.20, the pure unit quaternion vector $\mathbf{p}'$ and angle ϕ are

$$\mathbf{p}' = \frac{\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}}{\sqrt{14}} \quad \text{and} \quad \phi = \arccos\left(\frac{1}{\sqrt{15}}\right) = 1.3096\,\text{rad}.$$

Let us assume that vector $\mathbf{s}'$ to be rotated around the vector $\mathbf{p}'$ by angle ϕ is

$$\mathbf{s}' = 4\mathbf{i} - \mathbf{j} - 2\mathbf{k} \quad (|s'| = \sqrt{21}).$$

This vector is perpendicular to $\mathbf{p}'$, since

$$(p', s') = \frac{1}{\sqrt{15}}[1(4) + (-2)(-1) + 3(-2)] = 0.$$

Vector $\mathbf{s}'$ is rotated to vector $q\mathbf{s}'$, which can be calculated as

$$\begin{aligned} q\mathbf{s}' &= a\mathbf{s}' + \mathbf{q}' \times \mathbf{s}' = \tfrac{1}{\sqrt{15}}(4\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + \tfrac{1}{\sqrt{15}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 4 & -1 & -2 \end{vmatrix} \\ &= \tfrac{1}{\sqrt{15}}[(4\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + (7\mathbf{i} + 14\mathbf{j} + 7\mathbf{k})] = \tfrac{1}{\sqrt{15}}(11\mathbf{i} + 13\mathbf{j} + 5\mathbf{k}). \end{aligned}$$

We also can calculate this rotated vector by using multiplication in matrix form, as described in Eq. (1.37),

$$qs' = As' = \frac{1}{\sqrt{15}}\begin{pmatrix} 1 & -1 & 2 & -3 \\ 1 & 1 & -3 & -2 \\ -2 & 3 & 1 & -1 \\ 3 & 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \\ -1 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{15}}\begin{pmatrix} 0 \\ 11 \\ 13 \\ 5 \end{pmatrix}.$$

Here, vector $\mathbf{s}' = (0, s') = (0, 4, -1, -2)^T$. We can verify that the length of vector $q\mathbf{s}'$ equals $|s'| = \sqrt{21}$ by

$$|q\mathbf{s}'| = \frac{\sqrt{11^2 + 13^2 + 5^2}}{\sqrt{15}} = \frac{\sqrt{315}}{\sqrt{15}} = \frac{\sqrt{15 \cdot 21}}{\sqrt{15}} = \sqrt{21}.$$

1.3.3 The 2nd rotation: Multiplication by a perpendicular vector

The above described rotation by the angle ϕ is defined by the linear operator

$$q \to L_q(\mathbf{s}') = q\mathbf{s}', \tag{1.40}$$

for a given unit quaternion q, which is considered in the form of $q = \cos(\phi) + p' \sin(\phi)$ with $(p')^2 = -1$. Here, vectors $\mathbf{s}'$ are perpendicular to the imaginary part of q, i.e., $s' \perp \mathbf{q}'$, or $\mathbf{s}' \perp \mathbf{p}'$. To release this constraint and describe the rotation by any vector $\mathbf{s}'$ by operations of multiplication, we consider a modification of the operator L_q.

In quaternion algebra, the following operator of rotation is used:

$$q \to L_{q,\bar{q}}(\mathbf{s}') = q\mathbf{s}'q^{-1}, \tag{1.41}$$

where it is not necessary for vector $\mathbf{s}'$ to be perpendicular to $\mathbf{q}'$. We first consider the operator for a perpendicular vector $\mathbf{s}'$ and then use the fact that any vector $\mathbf{s}'$ can be presented as the sum of two vectors, one of which is perpendicular to $\mathbf{q}'$.

The operator $L_{q,\bar{q}}$ as the operator $L_{q,\bar{q}}$ preserves the length of the vector $\mathbf{s}'$. Indeed, one can see that

$$|L_{q,\bar{q}}(\mathbf{s}')| = |q\mathbf{s}'q^{-1}| = |q||s'||q^{-1}| = |s'|.$$

Since $|q| = 1$, the inverse quaternion q^{-1} is $\bar{q}$, i.e.,

$$q^{-1} = \bar{q} = \cos(\phi) - p' \sin(\phi). \tag{1.42}$$

Therefore,

$$\begin{aligned} L_{q,\bar{q}}(\mathbf{s}') &= (\mathbf{s}' \cos(\phi) + [\mathbf{p}' \times \mathbf{s}'] \sin(\phi))(\cos(\phi) - \mathbf{p}' \sin(\phi)) \\ &= \mathbf{s}' \cos^2(\phi) - \mathbf{s}'\mathbf{p}' \cos(\phi) \sin(\phi) + [\mathbf{p}' \times \mathbf{s}'] \sin(\phi) \cos(\phi) \\ &\quad - [\mathbf{p}' \times \mathbf{s}']\mathbf{p}' \sin^2(\phi). \end{aligned}$$

Vectors $\mathbf{s}'$ and $\mathbf{p}'$ are perpendicular; therefore, $\mathbf{s}'\mathbf{p}' = -[\mathbf{p}' \times \mathbf{s}']$, and vectors $[\mathbf{p}' \times \mathbf{s}']$ and $\mathbf{p}'$ are also perpendicular. This simplifies the above calculations as

$$\begin{aligned} L_{q,\bar{q}}(\mathbf{s}') &= \mathbf{s}' \cos^2(\phi) + 2[\mathbf{p}' \times \mathbf{s}'] \sin(\phi) \cos(\phi) + (\mathbf{s}'\mathbf{p}')\mathbf{p}' \sin^2(\phi) \\ &= \mathbf{s}' \cos^2(\phi) + 2[\mathbf{p}' \times \mathbf{s}'] \sin(\phi) \cos(\phi) + \mathbf{s}'(\mathbf{p}'\mathbf{p}') \sin^2(\phi) \\ &= \mathbf{s}' \cos^2(\phi) + 2[\mathbf{p}' \times \mathbf{s}'] \sin(\phi) \cos(\phi) - \mathbf{s}' \sin^2(\phi) \\ &= \mathbf{s}' \cos(2\phi) + [\mathbf{p}' \times \mathbf{s}'] \sin(2\phi), \end{aligned} \tag{1.43}$$

since $(\mathbf{p}')^2 = -1$. Thus, we obtain the rotation of vector $\mathbf{s}'$ around vector $\mathbf{p}'$, which is similar to the rotation in Eq. (1.39) but by the angle 2ϕ,

$$L_{q,\bar{q}}(\mathbf{s}') = q\mathbf{s}'\bar{q} = \mathbf{s}' \cos(2\phi) + \mathbf{v}' \sin(2\phi). \tag{1.44}$$

Such a rotation is illustrated in Fig. 1.18.

Example 1.22

We describe the rotation of the quaternion vector $\mathbf{s}' = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ around the unit vector

$$\mathbf{p}' = \frac{1}{3\sqrt{5}}(2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$$

by the angle $\varphi = 30°$. The required vector $\mathbf{v}'$ in Eq. (1.44) is calculated by

$$\begin{aligned} \mathbf{v}' = \mathbf{p}' \times \mathbf{s}' &= \frac{1}{3\sqrt{5}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 5 & 4 \\ 1 & 2 & -3 \end{vmatrix} = \frac{1}{3\sqrt{5}}[\mathbf{i}(-15-8) - \mathbf{j}(-6-4) + \mathbf{k}(4-5)] \\ &= \frac{1}{3\sqrt{5}}(-23\mathbf{i} + 10\mathbf{j} - \mathbf{k}). \end{aligned}$$

Therefore, the rotated vector $\mathbf{s}' \to \tilde{\mathbf{s}}'$ is calculated by

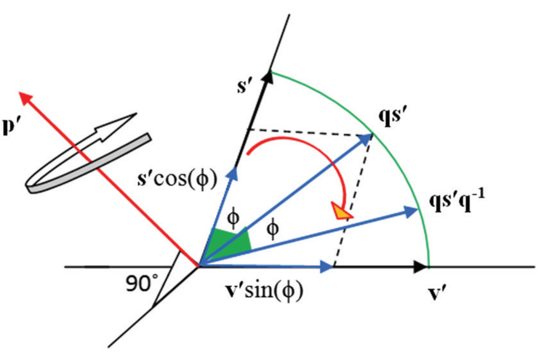

Figure 1.18 Rotation of vector $\mathbf{s}'$ around the unit vector $\mathbf{p}'$ by the angle 2ϕ.

$$\begin{aligned}\tilde{\mathbf{s}}' &= (\mathbf{i}+2\mathbf{j}-3\mathbf{k})\cos(30°)+\frac{1}{3\sqrt{5}}(-23\mathbf{i}+10\mathbf{j}-\mathbf{k})\sin(30°)\\ &= \mathbf{i}\left(\frac{\sqrt{3}}{2}-\frac{23}{6\sqrt{5}}\right)+\mathbf{j}\left(\sqrt{3}+\frac{\sqrt{5}}{3}\right)-\mathbf{k}\left(\frac{3\sqrt{3}}{2}+\frac{1}{6\sqrt{5}}\right)\\ &= -0.8483\mathbf{i}+2.4774\mathbf{j}-2.6726\mathbf{k},\end{aligned}$$

considering that $\cos(30°)=\sqrt{3}/2$ and $\sin(30°)=1/2$.

The vector $\tilde{\mathbf{s}}'$ is $q\mathbf{s}'\bar{q}$, but q has not been used directly in the above calculations. The quaternion q can be calculated by

$$\begin{aligned}q &= \cos\left(\frac{\varphi}{2}\right)+p'\sin\left(\frac{\varphi}{2}\right)=\cos(15°)+p'\sin(15°)\\ &= \cos(15°)+\frac{\sin(15°)}{3\sqrt{5}}(2i+5j+4k)=\frac{\sqrt{2+\sqrt{3}}}{2}+\frac{\sqrt{2-\sqrt{3}}}{2}\frac{2i+5j+4k}{3\sqrt{5}}\\ &= \frac{\sqrt{2+\sqrt{3}}}{2}+\frac{\sqrt{2-\sqrt{3}}}{6\sqrt{5}}(2i+5j+4k)=0.9659+0.0386(2i+5j+4k).\end{aligned}$$

1.3.4 Multiplication: Rotation of any vector

The operation of rotation by the angle 2ϕ can also be described for the general case of vector $\mathbf{s}'$. Indeed, let $\mathbf{s}'$ be the vector that is not perpendicular to $\mathbf{p}'$. It can be presented as the sum of two vectors

$$\mathbf{s}'=\mathbf{s}'_{\perp}+\mathbf{s}'_{\sim},$$

where vector $\mathbf{s}'_{\perp}$ is perpendicular to $\mathbf{p}'$, and $\mathbf{s}'_{\sim}$ is proportional to $\mathbf{p}'$; i.e., $\mathbf{s}'_{\sim}=\lambda\mathbf{p}'$ for a real number λ.

It is not difficult to notice that the multiplication $q(\lambda\mathbf{s}')$ is commutative. Indeed, the following calculations are valid:

$$q(\lambda\mathbf{p}')=(a+\mathbf{p}'|q'|)(\lambda\mathbf{p}')=\lambda(a\mathbf{p}'+\mathbf{p}'\mathbf{p}'|p'|)=\lambda\mathbf{p}'(a+p'|p'|)=(\lambda\mathbf{p}')q.$$

Therefore, vector $\mathbf{s}'_1$ remains invariable when applying the operator $L_{q,\bar{q}}$,

$$L_{q,\bar{q}}(\mathbf{s}'_{\sim})=L_{q,\bar{q}}(\lambda\mathbf{p}')=q(\lambda\mathbf{p}')\bar{q}=(\lambda\mathbf{p}')q\bar{q}=(\lambda\mathbf{p}')|q|^2=(\lambda\mathbf{p}')=\mathbf{s}'_{\sim}.$$

The operator $L_{q,\bar{q}}$ is linear; therefore, we obtain the following:

$$L_{q,\bar{q}}(\mathbf{s}')=L_{q,\bar{q}}(\mathbf{s}'_{\perp})+L_{q,\bar{q}}(\mathbf{s}'_{\sim})=L_{q,\bar{q}}(\mathbf{s}'_{\perp})+\mathbf{s}'_{\sim}. \tag{1.45}$$

The component $\mathbf{s}'_{\perp}$ is rotated around vector $\mathbf{p}'$, and another component remains invariable, which means that vector $\mathbf{s}'$ is rotated around $\mathbf{p}'$.

Statement 1.3 *Given pure unit quaternion q and vector $\mathbf{s}'$, the equation*

$$L_{q,\bar{q}}(\mathbf{s}') = q\mathbf{s}'\bar{q} = [\mathbf{s}'_{\perp}\cos(2\phi) + (p'\mathbf{s}'_{\perp})\sin(2\phi)] + \mathbf{s}'_{\sim} \tag{1.46}$$

describes the rotation of vector $\mathbf{s}'$ around vector $\mathbf{p}'$ by angle 2ϕ.

1.3.5 Matrix representation of rotation

We consider the operator of rotation $L_{q,\bar{q}}(\mathbf{s}') = q\mathbf{s}'\bar{q}$ in matrix form. The multiplication $q\mathbf{s}'$ can be calculated by matrix $\mathbf{A}_{L;q}$, i.e.,

$$q\mathbf{s}' = \mathbf{A}_{L;q}\mathbf{s}' = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}\begin{pmatrix} 0 \\ s_i \\ s_j \\ s_k \end{pmatrix}. \tag{1.47}$$

Here, vector $\mathbf{s}'$ is composed of its components $\mathbf{s}' = 0 + \mathbf{i}s_i + \mathbf{j}s_j + \mathbf{k}s_k$, and, similarly, $q\mathbf{s}'$ is a vector of $q\mathbf{s}'$.

We denote $\tilde{\mathbf{s}}' = (q\mathbf{s}')\bar{q}$ and the corresponding column vector by $\tilde{\mathbf{s}}'$. The multiplication of the number $q\mathbf{s}'$ from the right by $\bar{q}$ can be accomplished by matrix $\mathbf{A}_{R;\bar{q}}$, i.e.,

$$\tilde{\mathbf{s}}' = \mathbf{A}_{R;\bar{q}}(q\mathbf{s}') = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}(q\mathbf{s}').$$

Therefore, the rotation can be written as

$$\tilde{\mathbf{s}}' = \mathbf{A}_{R;\bar{q}}(q\mathbf{s}') = \mathbf{A}_{R;\bar{q}}(\mathbf{A}_{L;q}\mathbf{s}') = (\mathbf{A}_{R;\bar{q}}\mathbf{A}_{L;q})\mathbf{s}'. \tag{1.48}$$

The operator of rotation $L_{q,\bar{q}}$ is described by the matrix

$$\begin{aligned} \mathbf{A}_{\bar{q},q} = \mathbf{A}_{R;\bar{q}}\mathbf{A}_{L;q} &= \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 0 & 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 0 & 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{bmatrix}, \end{aligned} \tag{1.49}$$

considering that $a^2 + b^2 + c^2 + d^2 = |q|^2 = 1$.

We can also write this matrix of rotation as

$$\mathbf{A}_{\bar{q},q} = \mathbf{I}_4 + 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -(c^2+d^2) & (bc-ad) & (ac+bd) \\ 0 & (ad+bc) & -(b^2+d^2) & (cd-ab) \\ 0 & (bd-ac) & (ab+cd) & -(b^2+c^2) \end{bmatrix}, \tag{1.50}$$

where $\mathbf{I}_4$ is the 4×4 identity matrix.

In the particular case when q is a pure quaternion number, i.e., $a = 0$, this matrix is symmetric and equals

$$\mathbf{A}_{\bar{q},q} = \mathbf{I}_4 + 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -(c^2+d^2) & bc & bd \\ 0 & bc & -(b^2+d^2) & cd \\ 0 & bd & cd & -(b^2+c^2) \end{bmatrix}, \tag{1.51}$$

or

$$\mathbf{A}_{\bar{q},q} = \mathbf{I}_4 + 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-b^2 & bc & bd \\ 0 & bc & 1-c^2 & cd \\ 0 & bd & cd & 1-d^2 \end{pmatrix}, \tag{1.52}$$

since $|q|^2 = b^2 + c^2 + d^2 = 1$. One can notice that $\mathbf{A}_{\bar{q},q} = \mathbf{A}_{q,\bar{q}}$ and the determinant of this matrix $\det(\mathbf{A}_{\bar{q},q}) = 1$.

Example 1.23

Consider the rotation related to vector $s' = 4.5\mathbf{i} - 2\mathbf{j} + 3.5\mathbf{k}$ and the quaternion

$$q = \frac{1}{\sqrt{15}} + p'\frac{\sqrt{14}}{15} = \frac{1}{\sqrt{15}} + \frac{i - 2j + 3k}{\sqrt{14}}\frac{\sqrt{14}}{15} = \frac{1}{\sqrt{15}}(1 + i - 2j + 3k).$$

Vector $\mathbf{s}'$ can be presented as $\mathbf{s}' = \mathbf{s}'_\perp + \mathbf{s}'_\sim$, where

$$\mathbf{s}'_\perp = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{s}'_\sim = 0.5\mathbf{i} - \mathbf{j} + 1.5\mathbf{k} = 0.5(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}).$$

To calculate the rotated vector $L_{q,\bar{q}}(\mathbf{s}') = q\mathbf{s}'\bar{q}$, we compose the matrix of rotation $\mathbf{A}_{\bar{q},q}$. For a given q, we have $a = 1$, $b = 1$, $c = -2$, and $d = 3$ all up to the factor of $1/\sqrt{15}$. The matrix can be written as

$$\mathbf{A}_{\bar{q},q} = \frac{1}{15}\begin{pmatrix} 15 & 0 & 0 & 0 \\ 0 & -11 & -10 & 2 \\ 0 & 2 & -5 & -14 \\ 0 & 10 & -10 & 5 \end{pmatrix}, \quad [\det(\mathbf{A}_{\bar{q},q}) = 1]. \tag{1.53}$$

Therefore, the rotated vector is calculated as

$$\tilde{\mathbf{s}}' = \mathbf{A}_{\bar{q},q}\begin{pmatrix} 0 \\ 4 \\ -1 \\ 2 \end{pmatrix} + \mathbf{A}_{\bar{q},q}\begin{pmatrix} 0 \\ 0.5 \\ -1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.5 \\ -1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1.5 \\ -2 \\ 5.5 \end{pmatrix}.$$

Below is the script of the MATLAB-based code “example_one23.m” for this example with the function “rotateVarroundP.m,” which calculates the matrix $\mathbf{A}_{\bar{q},q}$ and rotated vector $\tilde{\mathbf{s}}'$.

```
%-------------------------------------------------------------
% example_one23.m   / Art Grigoryan, December 30, 2015
  q=[1,1,-2,3]/sqrt(15);
  s1=[0,4,-1,2];
  s2=[0,1,-2,3]/2;
  s=s1+s2; % 0   4.5 -2  3.5
  [qsq,matrixR] = rotateVarroundP(q,s);
  % qsq =      [ 0  -1.5 -2  5.5]
  [qsq1,matrixR] = rotateVarroundP(q,s1);
  % qsq1 =    [ 0  -2    -1    4]
  [qsq2,matrixR] = rotateVarroundP(q,s2);
  % qsq2 =    [ 0   .5   -1  1.5]
  %----------------- qsq = qsq1 + qsq2 -----------------
  R=round(15*matrixR)
  % 15   0    0    0
  %  0 -11  -10    2
  %  0   2   -5  -14
  %  0  10  -10    5
%-------------------------------------------------------------
% call: rotateVarroundP / Art Grigoryan, December 31, 2015
% Calculate the matrix of rotation and rotated vector s
% around the unit vector p' by the doubled angle 2*phi,
% where phi = acos(a) and q = a + p' sin(phi).
  function [qsq,matrixR] = rotateVarroundP(q,s)
    a=q(1); b=q(2); c=q(3); d=q(4);
    if size(s,1)==1 s=s'; end  % s to be a column-vector
    % 1st rotation
```

```
    A1=[a -b -c -d
         b  a -d  c
         c  d  a -b
         d -c  b  a];
    % 2nd rotation
    b=-b; c=-c; d=-d;  % for conjugate of quaternion q
    A2=[a -b -c -d
         b  a  d -c
         c -d  a  b
         d  c -b  a];
    % Matrix and Rotation by the doubled angle of arcos(a)
    matrixR=A2*A1;
    qsq=matrixR*s;          % as qs=A1*s; qsq=A2*qs;
%------------------------------------------------------
```

We also can formulate the task in the following way. Given unit vector $\mathbf{p}'$, rotate vector $\mathbf{s}'$ around $\mathbf{p}'$ by an angle 2ϕ. To use the above matrix of rotation, first the quaternion q should be calculated as

$$q = \cos(\phi) + p' \sin(\phi),$$

and then the matrix $\mathbf{A}_{\bar{q},q}$ can be constructed and used to calculate the rotated vector $L_{q,\bar{q}}(\mathbf{s}')$.

Example 1.24

Let vector $\mathbf{s}'$ be $4.5\mathbf{i} - 2\mathbf{j} + 3.5\mathbf{k}$ and rotated around the vector

$$\mathbf{p}' = \frac{1}{\sqrt{14}}(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

by the angle 2ϕ, where

$$\phi = \arccos(1/\sqrt{15}) = 1.3096\,\text{rad},$$

or 75.0368°. The quaternion q is calculated as

$$q = \cos(\phi) + p' \sin(\phi) = \frac{1}{\sqrt{15}} + \frac{i - 2j + 3k}{\sqrt{15}}.$$

As we know from Example 1.23, rotation of vector $\mathbf{s}'$ results in the vector

$$L_{q,\bar{q}}(\mathbf{s}') = -1.5\mathbf{i} - 2\mathbf{j} + 5.5\mathbf{k}.$$

Below is the script of the demo code “example_one24.m” for this example with the function “qsprotation.m,” which calculates the rotation of the vector s' around the vector p' by a given angle.

```
%------------------------------------------------------
% example_one24.m / Art Grigoryan, December 31, 2015
  s = [0, 4.5, -2, 3.5];
  p = [1,-2,3]/sqrt(14);
  phi_doubled=2*acos(1/sqrt(15));
  qsq = qsprotation(s,p,phi_doubled)
  % result: 0 -1.5 -2.0 5.5
%------------------------------------------------------
% qsprotation.m
% rotate a vector s around a unit vector p' by angle phi
% by using the operator with multiplication by quaternion
%     q = cos(phi) + p'sin(phi), (phi in radians).
  function qsq = qsprotation(s,p,phi_doubled)
    phi=phi_doubled/2;
    q=zeros(1,4);
    q(1)=cos(phi);
    q(2:4)=sin(phi)*p;
    qsq=rotateVarroundP(q,s);
%------------------------------------------------------
```

For this example, we have

$$\mathbf{A}_{q,\bar{q}} = \frac{1}{15}\begin{pmatrix} 15 & 0 & 0 & 0 \\ 0 & -11 & 2 & 10 \\ 0 & -10 & -5 & -10 \\ 0 & 2 & -14 & 5 \end{pmatrix}, = \mathbf{A}^T_{\bar{q},q},$$

and this property holds for all matrices $\mathbf{A}_{q,\bar{q}}$.

One can note that two quaternion numbers $q = a + ib + jc + kd$ and $-q = -a - ib - jc - kd$ define the same matrix, i.e.,

$$\mathbf{A}_{\bar{q},q} = \mathbf{A}_{-\bar{q},-q}.$$

Thus, up to the sign, the number q can be restored from the matrix $\mathbf{A} = \mathbf{A}_{\bar{q},q}$. We denote the elements of this matrix by $a_{n,m}$, $n, m = 0, 1, 2, 3$. If we add all elements of this matrix along the main diagonal, which is the trace, $\mathrm{tr}(\mathbf{A})$, we obtain the following (see Eq. 1.49):

$$\mathrm{tr}(\mathbf{A}) = a_{0,0} + a_{1,1} + a_{2,2} + a_{3,3} = 1 + 3a^2 - b^2 - c^2 - d^2 = 4a^2.$$

Thus, then the real part of q can be calculated by

$$a = \pm\frac{1}{2}\sqrt{\mathrm{tr}(\mathbf{A})}. \tag{1.54}$$

Three components of the imaginary part of q can be defined from other elements of the matrix as

$$a_{2,1} - a_{1,2} = 2ad, \quad a_{1,3} - a_{3,1} = 2ac, \quad a_{3,2} - a_{2,3} = 2ab,$$

and therefore

$$d = \frac{a_{2,1} - a_{1,2}}{2a}, \quad c = \frac{a_{1,3} - a_{3,1}}{2a}, \quad b = \frac{a_{3,2} - a_{3,2}}{2a}. \tag{1.55}$$

If the quaternion is pure, i.e., $a = 0$, we can use the second matrix-summand in Eq. (1.52), the elements of which we denote by $s_{n,m}$,

$$\mathbf{S}_{\bar{q},q} = (\{s_{n,m}\}_{n,m=0}^{3}) = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-b^2 & bc & bd \\ 0 & bc & 1-c^2 & cd \\ 0 & bd & cd & 1-d^2 \end{pmatrix}. \tag{1.56}$$

Three components of the imaginary part of q can be defined from the diagonal of this matrix as

$$s_{1,1} = 2(1-b^2), \quad s_{2,2} = 2(1-c^2), \quad a_{3,3} = 2(1-d^2).$$

Thus, up to the sign $\alpha = \pm 1$, we obtain the imaginary components:

$$b = \frac{\alpha\sqrt{1-s_{1,1}}}{2}, \quad c = \frac{\alpha\sqrt{1-s_{2,2}}}{2}, \quad d = \frac{\alpha\sqrt{1-s_{3,3}}}{2}. \tag{1.57}$$

1.3.6 Addition of rotations in 3-D space

We have used the property of linearity of the operation of rotation,

$$L_{q,\bar{q}}(\mathbf{s}' + \mathbf{v}') = L_{q,\bar{q}}(\mathbf{s}') + L_{q,\bar{q}}(\mathbf{v}')$$

for arbitrary vectors $\mathbf{s}'$ and $\mathbf{v}'$.

We also can consider the composition of the rotations, when a vector is rotated around an axis with unit vector $\mathbf{p}_1'$ and then around a different axis characterized by unit vector $\mathbf{p}_2'$ as

$$\mathbf{s}' \to L_{q_1,\bar{q}_1}(\mathbf{s}') \to L_{q_2,\bar{q}_2}[L_{q_1,\bar{q}_1}(\mathbf{s}')]. \tag{1.58}$$

Here, q_1 and q_2 are unit quaternions, which that can be written as

$$q_1 = \cos(\phi_1) + p_1' \sin(\phi_1) \quad \text{and} \quad q_2 = \cos(\phi_2) + p_2' \sin(\phi_2).$$

The quaternion multiplication is distributive, and therefore,

$$L_{q_2,\bar{q}_2}[L_{q_1,\bar{q}_1}(\mathbf{s}')] = L_{q_2,\bar{q}_2}(q_1\mathbf{s}'\bar{q}_1) = q_2(q_1\mathbf{s}'\bar{q}_1)\bar{q}_2 = (q_2q_1)\mathbf{s}'(\bar{q}_1\bar{q}_2),$$

and we know that $(\bar{q}_1\bar{q}_2) = \overline{q_2q_1}$. Therefore, denoting the quaternion q_2q_1 by

$$q = \cos(\phi) + p' \sin(\phi), \tag{1.59}$$

we obtain the following statement.

Statement 1.4 *The consequence of performing two rotations equivalent to the rotation around vector* $\mathbf{p}'$ *by angle* 2ϕ, *where both of these values can be easily calculated from the quaternion multiplication* $q = q_2 q_1$, *is*

$$L_{q_2,\bar{q}_2}[L_{q_1,\bar{q}_1}(\mathbf{s}')] = q\mathbf{s}'\bar{q} = L_{q,\bar{q}}(\mathbf{s}'). \tag{1.60}$$

Example 1.25

Consider two unit quaternions

$$q_1 = a_1 + q_1' = \frac{1}{\sqrt{7}} + \frac{i - j + 2k}{\sqrt{7}} \quad \text{and} \quad q_2 = a_2 + q_2' = \frac{1}{3} + \frac{2i - 2j}{3},$$

and the corresponding unit vectors

$$\mathbf{p}_1' = \frac{\mathbf{q}_1'}{|q_1'|} = \frac{\mathbf{i} - \mathbf{j} + 2\mathbf{k}}{\sqrt{6}} \quad \text{and} \quad \mathbf{p}_2' = \frac{\mathbf{q}_2'}{|q_2'|} = \frac{2\mathbf{i} - 2\mathbf{j}}{\sqrt{8}} = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}.$$

Let us assume that a vector $\mathbf{s}'$ is rotated first around vector $\mathbf{p}_1'$ and then around $\mathbf{p}_2'$ by angles $2\phi_1$ and $2\phi_2$, where

$$\phi_1 = \arccos\left(\frac{1}{\sqrt{7}}\right) = 67.7923° \quad \text{and} \quad \phi_2 = \arccos\left(\frac{1}{3}\right) = 70.5288°,$$

respectively. To find the resulting rotation, we first calculate the quaternion $q = q_2 q_1$. The real part of this number is

$$a = a_1 a_2 - (q_1', q_2') = \frac{1}{3\sqrt{7}}[1 - (2 + 2)] = -\frac{1}{\sqrt{7}},$$

and the imaginary part is

$$q' = (a_1 q_2' + a_2 q_1') + (q_1' \times q_2') = \tfrac{1}{3\sqrt{7}}[(2i - 2j) + (i - j + 2k)] - \tfrac{1}{3\sqrt{7}}\begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & -2 & 0 \end{vmatrix}$$
$$= \tfrac{1}{3\sqrt{7}}(3i - 3j + 2k) - \tfrac{1}{3\sqrt{7}}(4i + 4j) = \tfrac{1}{3\sqrt{7}}(-i - 7j + 2k).$$

The vector $\mathbf{q}'$ can be written as

$$\mathbf{q}' = \frac{-\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}}{3\sqrt{7}} = \frac{-\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}}{3\sqrt{6}} \cdot \sqrt{\frac{6}{7}}.$$

Therefore, the corresponding quaternion is written as

$$q = q_2 q_1 = -\frac{1}{\sqrt{7}} + \frac{-i - 7j + 2k}{3\sqrt{7}} = \frac{-3 - i - 7j + 2k}{3\sqrt{7}},$$

the angle of rotation is 2ϕ, where

$$\phi = \arccos(a) = \arccos\left(-\frac{1}{\sqrt{7}}\right) = 112.2077°,$$

and vector $\mathbf{p}'$ around which vector $\mathbf{s}'$ is to be rotated is

$$\mathbf{p}' = \frac{-\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}}{3\sqrt{6}}.$$

Thus, two consecutive rotations of vector $\mathbf{s}'$ by angles $2 \times 67.7923°$ and $2 \times 70.5288°$ around vectors $\mathbf{p}_1'$ and $\mathbf{p}_2'$ can be accomplished by one rotation of vector $\mathbf{s}'$ around vector $\mathbf{p}'$ by angle $2 \times 112.2077°$.

Example 1.26

In Example 1.25, the angles of rotations $2\phi_1$ and $2\phi_2$, as well as the summary angle 2ϕ, are large. Let us consider the consequence of rotating vector $\mathbf{s}'$ around the same unit vectors

$$\mathbf{p}_1' = \frac{\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}}{\sqrt{6}} \quad \text{and} \quad \mathbf{p}_2' = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$$

by 30° and 60°, respectively. Our task is to calculate the quaternion $q = q_2 q_1$ that describes this rotation.

Since

$$\cos(15°) = \frac{\sqrt{1 + \cos(30°)}}{2} = \frac{\sqrt{1 + \frac{\sqrt{3}}{2}}}{2} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

and

$$\sin(15°) = \frac{\sqrt{1 - \cos(30°)}}{2} = \frac{\sqrt{1 - \frac{\sqrt{3}}{2}}}{2} = \frac{\sqrt{2 - \sqrt{3}}}{2},$$

the quaternion $q_1 = a_1 + q_1'$ is calculated by

$$q_1 = \cos(15°) + p_1' \sin(15°) = \frac{\sqrt{2 + \sqrt{3}}}{2} + p_1' \frac{\sqrt{2 - \sqrt{3}}}{2}.$$

The second quaternion $q_2 = a_2 + q_2'$ is

$$q_2 = \cos(30°) + p_2' \sin(30°) = \frac{\sqrt{3}}{2} + p_2' \frac{1}{2}.$$

To calculate the vector $q = q_2 q_1$, we first consider its real part

$$\begin{aligned} a &= a_1 a_2 - (q_1', q_2') = a_1 a_2 - \frac{\sqrt{2-\sqrt{3}}}{2}\frac{1}{2}(p_1', p_2') \\ &= \frac{\sqrt{2+\sqrt{3}}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2-\sqrt{3}}}{4}\frac{1}{\sqrt{6}\sqrt{2}}(1+1+0) = \frac{\sqrt{6+3\sqrt{3}}}{4} - \frac{\sqrt{6-3\sqrt{3}}}{12}, \end{aligned}$$

or $a = 0.7618$. This number defines the doubled angle 2ϕ of rotation, where

$$\phi = \arccos(a) = 0.7047\,\text{rad},$$

or 40.3767°.

The imaginary part of $q = q_2 q_1$ can be calculated by

$$\begin{aligned} q' &= (a_2 q_1' + a_1 q_2') + q_2' \times q_1' = (a_2 q_1' + a_1 q_2') + \frac{\sqrt{2-\sqrt{3}}}{2}\frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{6}}\begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & -1 & 2 \end{vmatrix} \\ &= \left[\frac{\sqrt{3}}{2}\frac{\sqrt{2-\sqrt{3}}}{2}\cdot\frac{i-j+2k}{\sqrt{6}} + \frac{\sqrt{2+\sqrt{3}}}{2}\cdot\frac{i-j}{2\sqrt{2}}\right] + \frac{\sqrt{2-\sqrt{3}}}{8\sqrt{3}}(-2i-2j) \\ &= \left[\frac{\sqrt{4-2\sqrt{3}}}{8}(i-j+2k) + \frac{\sqrt{4+2\sqrt{3}}}{8}(i-j)\right] - \frac{\sqrt{6-3\sqrt{3}}}{12}(i+j). \end{aligned}$$

The components of this vector are

$$\begin{aligned} b &= \frac{\sqrt{4-2\sqrt{3}} + \sqrt{4+2\sqrt{3}}}{8} - \frac{\sqrt{6-3\sqrt{3}}}{12} = 0.3583, \\ c &= -\frac{\sqrt{4-2\sqrt{3}} + \sqrt{4+2\sqrt{3}}}{8} - \frac{\sqrt{6-3\sqrt{3}}}{12} = -0.5077, \\ d &= \frac{\sqrt{4-2\sqrt{3}}}{4} = 0.1830. \end{aligned}$$

Thus, the required quaternion is

$$q = a + q' = 0.7618 + (0.3583i - 0.5077j + 0.1830k), \tag{1.61}$$

and it can be verified that $|q|^2 = 1$. From this number, we find the unit vector

$$p' = \frac{q'}{|q'|} = \frac{0.3583i - 0.5077j + 0.1830k}{\sqrt{1-a^2}} = \frac{0.3583i - 0.5077j + 0.1830k}{0.6478}.$$

Of course, it is difficult to make such calculations by hand, and we can write the code to calculate all required data. As an example, a simple script of the code "example_one26.m" is given below with the input data of our example.

```
%-----------------------------------------------------------
% example_one26.m   / Art Grigoryan, January 1, 2016
% Rotation of the vector around two unit vectors p1 and p2
% by the given angles phi_1 and phi_2, respectively.
% 1. Given data:
  s = [0, 4.5, -2, 3.5];
  p1 = [1, -1,2]/sqrt(6); phi_1 = 30/180*pi;
  p2 = [1, -1,0]/sqrt(2); phi_2 = 60/180*pi;
% 2. Calculation of the quaternions q1, q2, and q = q2*q1:
  q1 = [cos(phi_1/2),sin(phi_1/2)*p1]; % 0.9659 0.1057
  -0.1057 0.2113
  q2 = [cos(phi_2/2),sin(phi_2/2)*p2]; % 0.8660 0.3536
  -0.3536 0
  q = mult2qs_direct(q2,q1);           % 0.7618 0.3583
  -0.5077 0.1830
% 3. Calculation of the vector p, angle, matrix, and rotated
  vector:
  [q_abs,p,phi] = qinpolform(q);       % 1, 0.5531 -0.7838
  0.2825, 0.7047
  [qsq,matrixR] = rotateVarroundP(q,s);
  %    qsq = [ -0.0000 0.9153 -4.2961 4.1480]
  % matrixR = [1        0        0        0
  %             0   0.4174 -0.6427 -0.6424
  %             0  -0.0850  0.6763 -0.7317
  %             0   0.9047  0.3601  0.2277]
%-----------------------------------------------------------
```

The results of this program are the following:

$$\begin{aligned} q_1 &= 0.9659 + 0.1057i - 0.1057j + 0.2113k, \\ q_2 &= 0.8660 + 0.3536i - 0.3536j, \\ q &= q_2 q_1 = 0.7618 + 0.3583i - 0.5077j + 0.1830k, \\ p' &= 0.5531i - 0.7838j + 0.2825k, \\ \phi &= 0.7047 \text{ rad.} \end{aligned} \tag{1.62}$$

Here, q coincides with the quaternion number in Eq. (1.61). The matrix of vector $\mathbf{s}'$ rotation around vector $\mathbf{p}'$ by the angle $\phi = 0.7047$, or 40.3767°, is

$$\mathbf{A}_{\bar{q},q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.4174 & -0.6427 & -0.6424 \\ 0 & -0.0850 & 0.6763 & -0.7317 \\ 0 & 0.9047 & 0.3601 & 0.2277 \end{pmatrix}, \quad [\det(\mathbf{A}_{\bar{q},q}) = 1],$$

and the rotated vector is

$$L_{q,\bar{q}}(\mathbf{s}') = 0.9153\mathbf{i} - 4.2961\mathbf{j} + 4.1480\mathbf{k}.$$

One can verify that, for this example, $|L_{q,\bar{q}}(\mathbf{s}')| = |\mathbf{s}'| = 6.0415$.

1.4 The Quaternion Exponential Function

The above-considered module $|q|$ of the quaternion is an example of the non-negative function over the quaternions $q = a + q'$. Other functions are the real component a and imaginary, or vector, component q' of the quaternion q, and the quaternion conjugate $\bar{q}$. In this section, we consider the concept of the exponential function in quaternion arithmetic, which is important in understanding rotations in the vector space of a quaternion, as well as in studying of the application of quaternion Fourier transforms, which will be described in the next chapters.

P1. We consider the Taylor series for the exponential function in the real case: [36]

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

together with two series

$$\cos(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$
$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

The first series for the imaginary number ix is

$$\begin{aligned}
e^{ix} &= \exp(ix) = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots \\
&= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] \\
&= \cos(x) + i\sin(x).
\end{aligned}$$

The Taylor series can also be used for the quaternions as

$$e^q = \exp(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n}{n!}. \tag{1.63}$$

The convergence of this series can be shown by considering two properties of the modulus operation: (1) the triangle inequality $|q_1 + q_2| \leq |q_1| + |q_2|$ and (2) modulus of the power $|q^n| = |q|^n$, for positive integer n. Indeed, the following inequalities hold:

$$|e^q| = \left|1 + \sum_{n=1}^{\infty} \frac{q^n}{n!}\right| \leq 1 + \sum_{n=1}^{\infty} \left|\frac{q^n}{n!}\right| = 1 + \sum_{n=1}^{\infty} \frac{|q^n|}{n!} = 1 + \sum_{n=1}^{\infty} \frac{|q|^n}{n!} = e^{|q|}.$$

Let us first describe the case when $q = \mu\vartheta$, where ϑ is a real number as an angle, and μ is a pure unit quaternion ($\mu^2 = -1$). One can note that

$$\mu^3 = (\mu^2)\mu = -\mu,\ \mu^4 = (\mu^2)^2 = 1,\ \mu^5 = (\mu^4)\mu = \mu,\ \mu^6 = (\mu^2)^3 = -1,\ \ldots$$

or

$$\mu^{2n} = (-1)^n, \quad \mu^{2n+1} = (-1)^n\mu, \quad n = 0, 1, 2, \ldots.$$

Therefore, the quaternion number $\exp(\mu\vartheta)$ can be written as

$$\begin{aligned}
e^{\mu\vartheta} &= 1 + \sum_{n=1}^{\infty} \frac{(\mu\vartheta)^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(\mu)^n(\vartheta)^n}{n!} \\
&= \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(\vartheta)^{2n}}{(2n)!}\right] + \mu\left[\sum_{n=0}^{\infty} (-1)^n \frac{(\vartheta)^{2n+1}}{(2n+1)!}\right] \\
&= \cos(\vartheta) + \mu\sin(\vartheta).
\end{aligned}$$

Any pure quaternion q can be written as

$$q = q' = \frac{q'}{|q'|}|q'| = \mu\vartheta, \quad \text{where} \quad \mu = \frac{q'}{|q'|},\ \vartheta = |q'| > 0,$$

and therefore,

$$e^q = \cos(\vartheta) + \mu\, \sin(\vartheta) = \cos(|q'|) + \frac{q'}{|q'|}\sin(|q'|).$$

In general, when $q = a + q'$, the real number a is commutative with q', i.e., $aq' = q'a$. Therefore, we can define the exponential function as

$$e^q = e^{a+q'} = e^a e^{q'} = e^a[\cos(\vartheta) + \mu\, \sin(\vartheta)], \tag{1.64}$$

or

$$e^q = e^a\left[\cos(|q'|) + \frac{q'}{|q'|}\sin(|q'|)\right].$$

We consider the exponent of the imaginary part of q to be

$$e^{q'} = \cos(\vartheta) + \mu\, \sin(\vartheta) = \cos(\vartheta) + i\mu_i \sin(\vartheta) + j\mu_j \sin(\vartheta) + k\mu_k \sin(\vartheta).$$

Therefore, the quaternion exponent can be written as

$$e^{q'} = [1 - (\mu_i + \mu_j + \mu_k)]\cos(\vartheta) + \mu_i e^{i\vartheta} + \mu_j e^{j\vartheta} + \mu_k e^{k\vartheta}.$$

It should be noted that, for a given pure quaternion μ, the number

$$Q = \cos(\vartheta) + \mu\, \sin(\vartheta)$$

is a quaternion number, and this number is the exponent $Q = e^{\mu\vartheta}$ only if μ is a pure unit quaternion. Otherwise, we can express the number

$$Q = (1 - |\mu|)\, \cos(\vartheta) + |\mu|\left[\cos(\vartheta) + \frac{\mu}{|\mu|}\sin(\vartheta)\right] = (1 - |\mu|)\, \cos(\vartheta) + |\mu| e^{\frac{\mu}{|\mu|}\vartheta}$$

as the arithmetic sum of the cosine and exponent.

Example 1.27

Consider the quaternion $q = 1 + i - 2j + k$, which can be written as

$$q = 1 + \mu\vartheta = 1 + \frac{i - 2j + k}{\sqrt{6}}\sqrt{6}.$$

Then, the exponential number $\exp(q)$ is calculated as

$$e^q = e^1\left[\cos(\sqrt{6}) + \frac{i - 2j + k}{\sqrt{6}}\sin(\sqrt{6})\right].$$

The real part of the exponent is

$$(e^q)_e = e^1 \cos(\sqrt{6}) = -2.0928,$$

and the components of the imaginary part equal

$$(e^q)_i = e^1 \frac{1}{\sqrt{6}} \sin(\sqrt{6}) = 0.7082,$$
$$(e^q)_j = e^1 \frac{-2}{\sqrt{6}} \sin(\sqrt{6}) = -1.4164,$$
$$(e^q)_k = e^1 \frac{1}{\sqrt{6}} \sin(\sqrt{6}) = 0.7082.$$

The exponent of q equals

$$e^q = -2.0928 + (0.7082i - 1.4164j + 0.7082k).$$

The length of this number $|e^q| = 2.7183$ and $e^{|q|} = e^{2.6458} = 14.0940$. One can note that, if the imaginary part has equal components, then the corresponding components of the imaginary part of the exponent $\exp(q)$ are equal, too. For instance, if $q_i = q_k$, then $(e^q)_i$ equals $(e^q)_k$; and if $q_i = q_j$, then $(e^q)_i = (e^q)_j$.

Below is the script of the MATLAB-based code "test_expofq.m" for this example with the function "expofq.m" to calculate the quaternion exponential function. We can also check that, for the numbers $q_1 = 1 + i$, $q_2 = 1 - 2j$, and $q_3 = 1 + k$, the exponential function equals

$$e^{1+i} = 1.4687 + 2.2874i,\ e^{1+k} = 1.4687 + 2.2874k,$$
$$e^{1-2j} = -1.1312 - 2.4717j,$$

and the complex exponential function $e^{1+i} = 1.4687 + 2.2874i$ and $e^{1-2i} = -1.1312 - 2.4717i$.

```
%-----------------------------------------------
% test_expof.m
  q=[1,1,-2,1];
  qexp = expofq(q);    % -2.0928 0.7082 -1.4164 0.7082
% Checking the function
  qexp=expofq([1,0,-2,0]);     % -1.1312 0 -2.4717 0
  zexp=exp(1-2*i);             % -1.1312 -2.4717i
  %
  qexp=expofq([1,1,0,0]);      % 1.4687 2.2874 0 0
  zexp=exp(1+i);               % 1.4687 + 2.2874i
  qexp=expofq([1,0,0,1]);      % 1.4687 0 0 2.2874
```

```
%--------------------------------------------------------------
% expofq.m / Artyom Grigoryan, January 8, 2016
% calculate the quaternion exponent function
  function qexp=expofq(q)
    qim=q(2:4);                 % imaginary part of q
    theta=norm(qim);            % angle theta=|qim|=sqrt
                                  (b^2+c^2+d^2)
    mu_vector=qim/theta;  % unit \mu
    e1=exp(q(1));
    qexp(1)=e1*cos(theta);
    qexp(2:4)=(e1*sin(theta))*mu_vector; % See Eq. 1.64
%--------------------------------------------------------------
```

P2. It is known that the cosine, sine, and complex exponential functions are periodic with the fundamental period $T = 2\pi$,

$$\cos(x) = \cos(x + 2\pi n), \quad \sin(x) = \sin(x + 2\pi n), \quad e^{ix} = e^{i(x+2\pi n)},$$

for any integer n. In the definition of the quaternion exponential function [Eq. (1.64)], the pure unit vector $\mu = q'/|q'|$ is used. This vector is directed as the vector q'; both vectors are on the same radial ray. Let us consider new vectors

$$q'_n = q' + \mu 2\pi n = q' + \frac{q'}{|q'|}2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots.$$

One can notice that, if two numbers q_1 and q_2 are commutative, i.e., $q_1 q_2 = q_2 q_1$, then $\exp(q_1 + q_2) = \exp(q_1)\exp(q_2)$. In our case, vectors q' and $(q'/|q'|)2\pi n$ are commutative; therefore,

$$e^{q'_n} = e^{q' + \frac{q'}{|q'|}2\pi n} = e^{q'} e^{\frac{q'}{|q'|}2\pi n} = e^{q'}\left[\cos(2\pi n) + \frac{q'}{|q'|}\sin(2\pi n)\right] = e^{q'}.$$

We also could write the following:

$$\begin{aligned} e^{q' + \frac{q'}{|q'|}2\pi n} &= e^{\frac{q'}{|q'|}(|q'|+2\pi n)} = \cos(|q'| + 2\pi n) + \frac{q'}{|q'|}\sin(|q'| + 2\pi n) \\ &= \cos(|q'|) + \frac{q'}{|q'|}\sin(|q'|) = e^{q'}. \end{aligned}$$

P3. Consider a pure unit quaternion number μ and the exponential number

$$\exp(\mu\vartheta) = \cos(\vartheta) + \mu\,\sin(\vartheta).$$

When the number $\mu = i, j$, or k, this exponential function is the traditional complex exponential function. Any unit quaternion number $q = a + q'$ can be written in such a form. Indeed, let $|q| = 1$, and μ be the pure unit quaternion $\mu = q'/|q'|$, i.e., $q' = \mu|q'|$. Because of the equality

$$a^2 + |q'|^2 = 1,$$

there exists an angle $\vartheta \in [0, \pi]$, such that

$$\cos(\vartheta) = a \quad \text{and} \quad \sin(\vartheta) = |q'|.$$

Therefore, we can write

$$q = a + \mu|q'| = \cos(\vartheta) + \mu \sin(\vartheta) = e^{\mu\vartheta.} \tag{1.65}$$

The unit vector μ is in the direction of vector q'.

P4. We now consider the inverse q^{-1} to the quaternion $q = \cos(\vartheta) + \mu \sin(\vartheta)$, i.e., such that $qq^{-1} = 1$. In complex arithmetic, the inverse to $z = \cos(\vartheta) + i \sin(\vartheta)$ is the complex conjugate $z^{-1} = \bar{z} = \cos(\vartheta) - i \sin(\vartheta)$. For the quaternion exponential function, a similar property holds:

$$q^{-1} = \bar{q} = \cos(\vartheta) - \mu \sin(\vartheta) = \exp(-\mu\vartheta). \tag{1.66}$$

Indeed, $q(\bar{q}) = |q|^2 = 1$, or we can perform the direct calculations as

$$[\cos(\vartheta) + \mu \sin(\vartheta)][\cos(\vartheta) - \mu \sin(\vartheta)] = \cos^2(\vartheta) - \mu^2 \sin^2(\vartheta) = 1,$$

where $\mu^2 = -|\mu|^2 = -1$.

Now we consider the inverse number to e^q, for any quaternion $q = a + \mu\vartheta$, where $\mu^2 = -1$. The following calculations hold:

$$\begin{aligned}(e^q)^{-1} &= (e^a e^{\mu\vartheta})^{-1} = e^{-a}(e^{\mu\vartheta})^{-1}\\ &= e^{-a}\frac{e^{-\mu\vartheta}}{|e^{\mu\vartheta}|^2} = e^{-a}e^{-\mu\vartheta} = e^{-a-\mu\vartheta} = e^{-q}.\end{aligned}$$

Thus,

$$(e^q)^{-1} = e^{-q} = e^{-a}[\cos(\vartheta) - \mu \sin(\vartheta)]. \tag{1.67}$$

P5. Consider two quaternions, $q_1 = 1 + 2i - j + k$ and $q_2 = 2 - i + 4j - 2k$. The exponents of these numbers and their sum are the following:

$$e^{q_1} = -2.0928 + (1.4164i - 0.7082j + 0.7082k),$$

$$e^{q_2} = -0.9565 + (1.5989i - 6.3954j + 3.1977k),$$

and

$$e^{(q_1+q_2)} = -19.7786 + (-1.0546i - 3.1638j + 1.0546k).$$

The product of two the exponents equals

$$e^{q_1}e^{q_2} = -7.0565 + (-2.4363i + 10.6650j - 15.2956k).$$

All of these numbers were calculated by the code "test_expof2.m," a script of which is given below.

```
%-----------------------------------------------------
% test_expof2.m
  q1 = [1, 2, - 1, 1];
  q2 = [2, - 1 ,4, - 2];
  q1exp = expofq(q1);      % −2.0928 1.4164 −0.7082 0.7082
  q2exp = expofq(q2);      % −0.9565 1.5989 −6.3954 3.1977
  q12exp = expofq(q1 + q2); % −19.7786 −1.0546 −3.1638
                             1.0546
  qexp12=mult2qs_direct(q1exp,q2exp);
                           % −7.0565 −2.4363 10.6650
                             − 15.2956
%-----------------------------------------------------
```

One can see that, for these quaternions,

$$e^{(q_1+q_2)} \neq e^{q_1}e^{q_2}.$$

This statement is also valid in the general case when $q_1 \neq kq_2$, where k is a real number.

P6. It is not difficult to see that, for any angles ϑ_1 and ϑ_2, the following holds:

$$\begin{aligned}\exp(\mu\vartheta_1)\exp(\mu\vartheta_2) &= [\cos(\vartheta_1) + \mu\ \sin(\vartheta_1)][\cos(\vartheta_2) + \mu\ \sin(\vartheta_2)]\\ &= [\cos(\vartheta_1)\ \cos(\vartheta_2) + \mu^2 \sin(\vartheta_1)\ \sin(\vartheta_2)]\\ &\quad + \mu[\cos(\vartheta_1)\ \sin(\vartheta_2) + \sin(\vartheta_1)\ \cos(\vartheta_2)]\\ &= \cos(\vartheta_1 + \vartheta_2) + \mu\ \sin(\vartheta_1 + \vartheta_2) = \exp[\mu(\vartheta_1 + \vartheta_2)].\end{aligned}$$

In the $\vartheta_1 = \vartheta_2$ case, we obtain

$$(e^{\mu\vartheta_1})^2 = e^{2\mu\vartheta_1};$$

therefore, $(e^q)^2 = e^{2q}$ for any quaternion number.

For any pure unit quaternions μ and ν,

$$\exp(\mu\vartheta_1)\exp(\nu\vartheta_1) = [\cos^2(\vartheta_1) + \mu\nu\ \sin^2(\vartheta_1)] + \frac{\mu+\nu}{2}\sin(2\vartheta_1);$$

therefore, if $\mu \neq \nu$,

$$\exp(\mu\vartheta_1)\exp(\nu\vartheta_1) \neq \exp[(\mu+\nu)\vartheta_1].$$

P7. The quaternion also can be represented in a polar form as

$$q = |q|\exp(\mu\vartheta),$$

where μ is a pure unit quaternion $\mu = i\mu_i + j\mu_j + k\mu_k$, such that $|\mu| = 1$, $\mu^2 = -1$, and ϑ is a real angle in the interval $[0, \pi]$ Indeed,

$$\begin{aligned} q = a + q' = |q|\left(\frac{a}{|q|} + \frac{q'}{|q|}\right) &= |q|\left(\frac{a}{|q|} + \frac{q'}{|q'|} + \frac{|q'|}{|q|}\right) \\ = |q|\left(\frac{a}{|q|} + \mu\frac{|q'|}{|q|}\right) &= |q|[\cos(\vartheta) + \mu\ \sin(\vartheta)], \end{aligned}$$

where the pure unit quaternion μ and the angle ϑ are calculated by

$$\mu = \frac{q'}{|q'|} \quad \text{and} \quad \vartheta = \arccos\left(\frac{a}{|q|}\right). \tag{1.68}$$

It should be noted that the sine of this angle is not negative, $\sin(\vartheta) = |q'|/|q|$ therefore, the angles ϑ in the polar form of the quaternions are from the interval $[0, \pi]$.

The polar form is unique, as the classic polar form of complex numbers. Indeed, consider the following equation for the unit quaternion q with $|q| = 1$:

$$q = qe + ib + jc + kd = \cos(\vartheta) + (i\mu_i + j\mu_j + k\mu_k)\ \sin(\vartheta),$$

from which it follows that if $|q_e| \neq 1$; then,

$$\cos(\vartheta) = a, \mu_i = b/\sin(\vartheta), \mu_j = c/\sin(\vartheta), \mu_k = d/\sin(\vartheta).$$

In the $|a| = 1$ case, $b = c = d = 0$; therefore, $\mu_i = \mu_j = \mu_k = 0$ and $\vartheta = \pi$.

P8. Given a real number x, the power of quaternion q is defined as

$$\begin{aligned} q^x = [|q|\exp(\mu\vartheta)]^x &= |q|^x[\exp(\mu\vartheta)]^x \\ &= |q|^x[\exp(\mu x\vartheta)] = |q|^x[\cos(x\vartheta) + \mu\ \sin(x\vartheta)]. \end{aligned}$$

We can verify this definition for the square of q as follows:

$$\begin{aligned} |q|^2[\cos(2\vartheta)] + \mu\ \sin(2\vartheta)] &= |q|^2[\cos^2(\vartheta) - \sin^2(\vartheta) + 2\mu\ \sin(\vartheta)\ \cos(\vartheta)] \\ = |q|^2[\cos(\vartheta) + \mu\ \sin(\vartheta)]^2 &= \{|q|[\cos(\vartheta) + \mu\ \sin(\vartheta)]\}^2 = (q)^2. \end{aligned}$$

The $x = 1/2$ case corresponds to the calculation of the square root of the quaternion q,

$$\sqrt{q} = q^{1/2} = \sqrt{|q|}\left[\cos\left(\frac{\vartheta}{2}\right) + \mu\ \sin\left(\frac{\vartheta}{2}\right)\right].$$

Example 1.28

Consider the square root, square, and cube of the quaternion $q = 1 - i + 2j + 3k$ with the length $|q| = \sqrt{1+1+4+9} = \sqrt{15}$, which can be written as

$$q = \sqrt{15}\left(\frac{1}{\sqrt{15}} + \frac{-i+2j+3k}{\sqrt{15}}\right),$$

and whose imaginary part is

$$q' = -i + 2j + 3k = \frac{-i+2j+3k}{\sqrt{14}} \cdot \sqrt{14}.$$

The angle ϑ is calculated by

$$\vartheta = \arccos\left(\frac{1}{\sqrt{15}}\right) = 75.0368°,$$

and the vector is

$$\mu = \frac{q'}{|q'|} = \frac{-i+2j+3k}{\sqrt{14}}.$$

The following calculations hold for the square root:

$$\begin{aligned}\sqrt{q} &= \sqrt{|q|}\left[\cos\left(\frac{\vartheta}{2}\right) + \mu\ \sin\left(\frac{\vartheta}{2}\right)\right]\\ &= \sqrt{\sqrt{15}}\left[\cos(37.5184°) + \frac{-i+2j+3k}{\sqrt{14}}\sin(37.5184°)\right]\\ &= 1.5609 + 0.3203(-i+2j+3k).\end{aligned}$$

The square of q can be calculated by

$$\begin{aligned}q^2 &= |q|^2[\cos(2\vartheta) + \mu\ \sin(2\vartheta)] = 15[\cos(2\vartheta) + \mu\ \sin(2\vartheta)]\\ &= 15\left[\cos(150.0736°) + \frac{-i+2j+3k}{\sqrt{14}}\sin(150.0736°)\right]\\ &= -13 + 2(-i+2j+3k).\end{aligned}$$

The cube of q can be calculated similarly;

$$\begin{aligned} q^3 &= |q|^3[\cos(3\vartheta) + \mu \sin(3\vartheta)] \\ &= (\sqrt{15})^3\left[\cos(225.1103°) + \frac{-i+2j+3k}{\sqrt{14}}\sin(225.1103°)\right] \\ &= -41 - 11(-i+2j+3k). \end{aligned}$$

P9. Using the polar form of the quaternion $q = a + (ib + jc + kd)$, we can define the quaternion logarithm function. Indeed, we can write q as

$$q = |q|e^{\mu\vartheta} = e^{\ln|q|}e^{\mu\vartheta} = e^{\ln|q|+\mu\vartheta}.$$

Then, the logarithm of q can be defined as

$$\ln q = \ln|q| + \mu\vartheta = \ln|q| + \frac{ib+jc+kd}{\sqrt{b^2+c^2+d^2}}\arccos\left(\frac{a}{|q|}\right). \tag{1.69}$$

This is a quaternion number, which we denote by

$$\ln q = a_1 + ib_1 + jc_1 + kd_1$$

with the components

$$\begin{aligned} a_1 &= \ln|q| = \frac{1}{2}\ln(\sqrt{a^2+b^2+c^2+d^2}), \\ b_1 &= \frac{b}{\sqrt{b^2+c^2+d^2}}\arccos\left(\frac{a}{|q|}\right), \\ c_1 &= \frac{c}{\sqrt{b^2+c^2+d^2}}\arccos\left(\frac{a}{|q|}\right), \\ d_1 &= \frac{d}{\sqrt{b^2+c^2+d^2}}\arccos\left(\frac{a}{|q|}\right). \end{aligned} \tag{1.70}$$

Thus, the components b_1, c_1 , and d_1 are proportional to the components $b, c,$ and d, respectively:

$$\frac{b_1}{b} = \frac{c_1}{c} = \frac{d_1}{d} = \frac{1}{|q'|}\arccos\left(\frac{a}{|q|}\right).$$

Example 1.29

Consider the quaternion $q = 1 + 2i - 3j + k$. The logarithm of this number is

$$\ln q = 1.3540 + 0.700i - 1.0500j + 0.3500k.$$

Indeed, $|q| = \sqrt{1+4+9+1} = \sqrt{15} = 3.8730$, $a_1 = \ln 3.8730 = 1.3540$, and the imaginary components are

$$b_1 = \frac{2}{\sqrt{14}} \arccos\left(\frac{1}{\sqrt{15}}\right) = \frac{2}{\sqrt{14}} \cdot 1.3096 = 0.7000,$$

$$c_1 = \frac{-3}{\sqrt{14}} \arccos\left(\frac{1}{\sqrt{15}}\right) = -b_1 \frac{3}{2} = -1.0500,$$

$$d_1 = \frac{1}{\sqrt{14}} \arccos\left(\frac{1}{\sqrt{15}}\right) = \frac{b_1}{2} = 0.3500.$$

Below is the script of the MATLAB-based code for this example, "test_logofq.m," which uses the function "logofq.m" to calculate the logarithm by Eq. (1.70). The command "qexp = expofq(qq)" is used to verify that the exponent of the logarithm of q is q.

```
%-----------------------------------------------------------
% test_logofq.m
  q=[1,2,-3,1];
  qq=logofq(q);          % 1.3540 0.7000 -1.0500 0.3500
  qexp=expofq(qq);       % is q=[1 2 -3 1];
%-----------------------------------------------------------
% call: logofq.m / Art Grigoryan, January 9, 2016
% calculate the quaternion exponent function
%-----------------------------------------------------------
  function qln=logofq(q)
    q_modul=norm(q);
    a1=log(q_modul);
    qim=q(2:4);            % imaginary part of q
    qm=norm(qim);          % qm=sqrt(b^2+c^2+d^2)
    phi=acos(q(1)/q_modul);
    k=phi/qm;
    bcd1=k*q(2:4);         % imaginary part [b1 c1 d1]
    qln=[a1 bcd1];         % ln(q)=[a1 b1 c1 d1]
%-----------------------------------------------------------
```

To express the components of the number q by the components of the number $\ln q$, we can represent $q = a + bi + cj + dk$ as the logarithm in the exponent

$$q = e^{\ln q} = e^{a_1+ib_1+jc_1+kd_1} = e^{a_1} e^{ib_1+jc_1+kd_1}.$$

We consider all components of the quaternion exponent,

$$q = e^{a_1} e^{ib_1 + jc_1 + kd_1} = e^{a_1} e^{\frac{ib_1 + jc_1 + kd_1}{\sqrt{b_1^2 + c_1^2 + d_1^2}} \sqrt{b_1^2 + c_1^2 + d_1^2}}$$
$$= e^{a_1} \left[\cos\left(\sqrt{b_1^2 + c_1^2 + d_1^2} \right) + \frac{ib_1 + jc_1 + kd_1}{\sqrt{b_1^2 + c_1^2 + d_1^2}} \sin \left(\sqrt{b_1^2 + c_1^2 + d_1^2} \right) \right].$$

Let $\varphi = \sqrt{b_1^2 + c_1^2 + d_1^2}$. Then, we can write that the real part of q is

$$a = e^{a_1} \cos(\varphi), \text{ and } a_1 = \ln|q|. \tag{1.71}$$

To express the imaginary components of the logarithm by the number q, we use the following equalities:

$$b = \frac{b_1|q|}{\varphi} \sin(\varphi), \quad c = \frac{c_1|q|}{\varphi} \sin(\varphi), \quad d = \frac{d_1|q|}{\varphi} \sin(\varphi).$$

Using the sinc function

$$\sin(\varphi) = \frac{\sin(\varphi)}{\varphi}, \quad \text{sinc}(0) = 1,$$

we can write

$$b = b_1|q|\text{sinc}(\varphi), \quad c = c_1|q|\text{sinc}(\varphi), \quad d = d_1|q|\text{sinc}(\varphi). \tag{1.72}$$

Therefore, the components of the quaternion logarithm can also be calculated by

$$\begin{aligned} b_1 &= \frac{b}{|q|} [\text{sinc}(\varphi)]^{-1}, \\ c_1 &= \frac{c}{|q|} [\text{sinc}(\varphi)]^{-1}, \quad \text{or} \quad c_1 = c\frac{b_1}{b} \text{ if } b \neq 0, \\ d_1 &= \frac{d}{|q|} [\text{sinc}(\varphi)]^{-1}, \quad \text{or} \quad d_1 = d\frac{b_1}{b} \text{ if } b \neq 0. \end{aligned} \tag{1.73}$$

Below is the script of the code “test_logofq2.m” for the calculation of the logarithm of the quaternion $q = 1 + 2i - 3j + k$. In this script, the function “logofq2.m” is used to calculate the logarithm of the quaternion by Eq. (1.73).

```
%-----------------------------------------------------------
% test_logofq2.m
  q=[1,2,-3,1];
  qq=logofq2(q);         % 1.3540 0.7000 -1.0500 0.3500
  qexp = expofq(qq);     % is q = [1 2 -3 1];
%-----------------------------------------------------------
```

```
% call: logofq2.m / Art Grigoryan, January 9, 2016
% calculate the quaternion exponent function
 function qln = logofq2(q)
   q_modul=norm(q);
   a1=log(q_modul);
   phi=acos(q(1)/q_modul);
   k=q_modul*sinc(phi/pi);    % See Eq. 1.72
   bcd1 = q(2:4)/k;           % imaginary part [b1 c1 d1]
   qln=[a1 bcd1];             % ln(q) = [a1 b1 c1 d1]
%--------------------------------------------------
```

Thus, two functions "logofq(q)" and "logofq2(q)" can be used to calculate the logarithm of a quaternion number q. We note here that the command "sinc" in MATLAB calculates sinc(φ) as $\sin(\pi\varphi)/[\pi(\varphi)]$.

1.5 Quaternion Trigonometric and Hyperbolic Functions

The quaternion exponential function allows for extending the concept of the cosine, sine, and tangent functions to the quaternion space. Given a pure unit quaternion number μ, the exponential function of angle ϑ is defined as

$$e^{\mu\vartheta} = \cos(\vartheta) + \mu \sin(\vartheta),$$

and $e^{-\mu\vartheta} = \cos(\vartheta) - \mu \sin(\vartheta)$. Therefore, the real cosine and sine functions can be written as

$$\cos(\vartheta) = \frac{1}{2}(e^{\mu\vartheta} + e^{-\mu\vartheta}),$$

and

$$\sin(\vartheta) = \frac{1}{2\mu}(e^{\mu\vartheta} - e^{-\mu\vartheta}) = \frac{\overline{\mu}}{2}(e^{\mu\vartheta} - e^{-\mu\vartheta}).$$

To extend these concepts to the quaternion functions, we define the quaternion cosine and sine in a way that is similar to the complex case:

$$\cos(q) = \frac{1}{2}(e^{\mu q} + e^{-\mu q}), \quad \sin(q) = \frac{1}{2\mu}(e^{\mu q} - e^{-\mu q}).$$

Here, the unit μ can be considered to be the number from the representation of the quaternion $q = a_1 + \mu\vartheta$.

The quaternion hyperbolic cosine and sine can be defined as

$$\cosh(q) = \frac{1}{2}(e^{q} + e^{-q}), \quad \sinh(q) = \frac{1}{2}(e^{q} - e^{-q}).$$

Example 1.30

Consider the quaternion $q = 1 + 2i + \sqrt{3}j - 3k$ and write it as

$$q = 1 + \frac{2i + \sqrt{3}j - 3k}{\sqrt{4+3+9}} \cdot 4.$$

When representing this quaternion as $q = a + \mu\vartheta$, we have $a = 1, \vartheta = 4$, and

$$\mu = \frac{2i + \sqrt{3}j - 3k}{\sqrt{4}} = \frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k, \quad (\mu^2 = -1).$$

Therefore, the exponent $\exp(q)$ can be written as

$$e^q = e^a e^{\mu\vartheta} = e^{-1}\left[\cos(4) + \left(\frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k\right)\ \sin(4)\right],$$

and

$$e^{-q} = e^{-a} e^{-\mu\vartheta} = e^{-1}\left[\cos(4) + \left(\frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k\right)\ \sin(4)\right].$$

The number μq is $\mu(a + \mu\vartheta) = -\vartheta + a\mu$; therefore,

$$\begin{aligned} e^{\mu q} &= e^{-\vartheta} e^{a\mu} = e^{-\vartheta}[\cos(a) + \mu\ \sin(a)] \\ &= e^4\left[\cos(1) + \left(\frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k\right)\ \sin(1)\right], \end{aligned}$$

and

$$e^{-\mu q} = e^{-\vartheta} e^{-a\mu} = e^4\left[\cos(1) - \left(\frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k\right)\ \sin(1)\right].$$

Therefore, the cosine of the quaternion q can be calculated as follows:

$$\cos(q) = \frac{1}{2}(e^{\mu q} + e^{-\mu q}) = \cos(1) \cdot \frac{e^{-4} + e^4}{2} + \frac{e^{-4} - e^4}{2}\left(\frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k\right)\ \sin(1).$$

This cosine can also be written as

$$\cos(q) = \cos(1)\cosh(4) - \mu\ \sinh(4)\ \sin(1).$$

It also not difficult to see that the quaternion sine can be calculated by

$$\sin(q) = \sin(1)\cosh(4) + \mu\ \sinh(4)\ \cos(1).$$

Indeed, since $\mu^2 = -1$, we can write

$$\begin{aligned}
\frac{1}{2\mu}\left(e^{\mu q} - e^{-\mu q}\right) &= \cos(1) \cdot \frac{e^{-4} - e^{4}}{2\mu} + \frac{e^{-4} + e^{4}}{2\mu}\left(\frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k\right)\sin(1)\\
&= \cos(1) \cdot \frac{e^{-4} - e^{4}}{2\mu} + \frac{e^{-4} + e^{4}}{2}\sin(1)\\
&= \cos(1) \cdot \frac{e^{-4} - e^{4}}{-2}\mu + \frac{e^{-4} + e^{4}}{2}\sin(1)\\
&= \cos(1)\sinh(4)\mu + \cosh(4)\,\sin(1).
\end{aligned}$$

The quaternion hyperbolic cosine can be calculated by

$$\cosh(q) = \frac{1}{2}\left(e^{q} + e^{-q}\right) = \cos(4) \cdot \frac{e^{1} + e^{-1}}{2} + \frac{e^{1} - e^{-1}}{2}\left(\frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k\right)\sin(4),$$

or

$$\cosh(q) = \cos(4)\,\cosh(1) + \left(\frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k\right)\sinh(1)\,\sin(4).$$

The quaternion hyperbolic sine can similarly be calculated as

$$\sinh(q) = \frac{1}{2}\left(e^{q} - e^{-q}\right) = \cos(4) \cdot \frac{e^{1} - e^{-1}}{2} + \frac{e^{1} + e^{-1}}{2}\left(\frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k\right)\sin(4),$$

or

$$\sinh(q) = \cos(4)\sinh(1) + \left(\frac{1}{2}i + \frac{\sqrt{3}}{4}j - \frac{3}{4}k\right)\cosh(1)\,\sin(4).$$

Summarizing the above calculations, we can write that the quaternion cosine and sine and hyperbolic cosine and sine can be represented as

$$\begin{aligned}
\cos(q) &= \cos(a)\cosh(\vartheta) - \mu\,\sinh(\vartheta)\,\sin(a),\\
\sin(q) &= \sin(a)\cosh(\vartheta) + \mu\,\sinh(\vartheta)\,\cos(a),\\
\cosh(q) &= \cos(\vartheta)\cdot\cosh(a) + \mu\,\sinh(a)\,\sin(\vartheta),\\
\sinh(q) &= \cos(\vartheta)\cdot\sinh(a) + \mu\,\cosh(a)\,\sin(\vartheta),
\end{aligned} \tag{1.74}$$

This representation is valid for any quaternion q.

The hyperbolic tangent is defined as

$$\tanh(q) = \frac{\sinh(q)}{\cosh(q)} = \frac{e^{q} - e^{-q}}{e^{q} + e^{-q}} = \frac{e^{q} - e^{-q}}{e^{q} + e^{-q}} \cdot \frac{e^{q}}{e^{q}} = \frac{e^{2q} - 1}{e^{2q} + 1},$$

since according to Eq. (1.67), $e^{-q} = (e^{q})^{-1}$, or $e^{-q}e^{q} = 1$.

Here, we note that in the division of the hyperbolic sine, the left and right divisions by the hyperbolic cosine are the same, i.e.,

$$\frac{\sinh(q)}{\cosh(q)} = \frac{1}{\cosh(q)}\sinh(q) = \sinh(q)\frac{1}{\cosh(q)}.$$

Indeed, as mentioned in sub-section 1.1.5 (property P4), when dividing quaternions q_1 by q_2, the divisions from the right and left are the same if $\overline{q}_2 q_1 = q_1 \overline{q}_2$. In our case, this requirement holds:

$$\overline{(e^q + e^{-q})}(e^q - e^{-q}) = (e^{-q} + e^q)(e^q - e^{-q}) = e^{2q} - e^{12q}$$

and

$$(e^q - e^{-q})\overline{(e^q + e^{-q})} = (e^q - e^{-q})(e^{-q} + e^q) = e^{2q} - e^{-2q}.$$

The hyperbolic cotangent is similarly defined as

$$\coth(q) = \frac{\cosh(q)}{\sinh(q)} = \frac{e^q + e^{-q}}{e^q - e^{-q}} = \frac{e^q + e^{-q}}{e^q - e^{-q}} \cdot \frac{e^q}{e^q} = \frac{e^{2q} + 1}{e^{2q} - 1}.$$

1.6 Quaternion-Type Numbers

In 4-D space, numbers other than quaternion numbers can be considered and applied in signal and image processing. We mention three such types of numbers, which are called *pseudo-quaternions*, *degenerate quaternions*, and *degenerate pseudo-quaternions* [34].

- Pseudo-quaternions $q = (a, b, c, d)$ are defined as the numbers

$$q = a + ib + ec + fd,$$

where i,e, and f are unit numbers with multiplications given in Table 1.3.
The conjugate number $\overline{q}$ is defined as

$$\overline{q} = a - ib - ec - fd;$$

therefore, the module of the pseudo-quaternion number q is defined by

Table 1.3 Table $T(1, i, e, f)$ of multiplications of pseudo-quaternion unit numbers.

	1	i	e	f
1	1	i	e	f
i	i	-1	f	$-e$
e	e	$-f$	1	$-i$
f	f	e	i	1

$$|q|^2 = |q\overline{q}| = a^2 - b^2 - c^2 - d^2.$$

- The degenerate quaternions $q = (a, b, c, d)$ are defined as the numbers

$$q = a + ib + \varepsilon c + \eta d,$$

where i, ε, and η are the numbers with multiplications given in Table 1.4.
The conjugate number $\overline{q}$ is defined as

$$\overline{q} = a - ib - \varepsilon c - \eta d.$$

The module of the degenerate quaternion number q is defined by

$$|q|^2 = |q\overline{q}| = a^2 + b^2.$$

- The degenerate pseudo-quaternions $q = (a, b, c, d)$ are defined as the numbers

$$q = a + eb + \varepsilon c + \zeta d,$$

where e, ε, and ζ are the numbers with multiplications given in Table 1.5.
The conjugate number $\overline{q}$ is defined as

$$\overline{q} = a - ib - \varepsilon c - \xi d.$$

The module of the degenerate quaternion number q is defined by

$$|q|^2 = |q\overline{q}| = a^2 - b^2.$$

Table 1.4 Table $T(1, i, \varepsilon, \eta)$ of multiplications of degenerate quaternion unit numbers.

	1	i	ε	η
1	1	i	ε	η
i	i	-1	η	$-\varepsilon$
ε	ε	$-\eta$	0	0
η	η	ε	0	0

Table 1.5 Table $T(1, e, \varepsilon, \zeta)$ of multiplications of degenerate pseudo-quaternion unit numbers.

	1	e	ε	ζ
1	1	e	ε	ζ
e	e	1	ζ	ε
ε	ε	$-\zeta$	0	0
ζ	ζ	$-\varepsilon$	0	0

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