Comparison of matrix method and ray tracing in the study of complex optical systems

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ABSTRACT

In the context of the classical study of optical systems within the geometrical Gauss approximation, the cardinal elements are efficiently obtained with the aid of the transfer matrix between the input and output planes of the system. In order to take into account the geometrical aberrations, a ray tracing approach, using the Snell-Descartes laws, has been implemented in an interactive software. Both methods are applied for measuring the correction to be done to a human eye suffering from ametropia. This software may be used by optometrists and ophthalmologists for solving the problems encountered when considering this pathology. The ray tracing approach gives a significant improvement and could be very helpful for a better understanding of an eventual surgical act.

Keywords: Matrix method, cardinal elements, ray tracing, eye.

1. INTRODUCTION

This article is devoted to the study of complex optical systems in the framework of geometrical optics. The ray optics is the branch of optics in which all the wave effects are neglected: the light is considered as travelling along rays which can only change direction by refraction or reflection. On one hand, a further simplifying approximation can be made if attention is restricted to rays travelling close to the optical axis and at small angles: the well-known linear or paraxial approximation introduced by Gauss. On the other hand, in order to estimate the geometrical aberrations, it is sometimes necessary to pay attention to marginal rays. Both methods are presented and applied to the study of the human eye suffering from ametropia. Thus, we intend to show that the tracing of both rays, paraxial and marginal, is necessary for measuring the correction to be done. This work has lead on the development of an interactive software that may be used by optometrists and ophthalmologists for solving the problems encountered when considering certain pathologies of the eye.

Within the frame of the linear approximation of geometrical optics, the properties of the rays travelling in a medium or through an optical system can be treated with an elegant and powerful matrix formalism. When using optical angles rather than geometrical ones, the elementary matrices involved in the propagation have a nice property: their determinant is equal to unity. This formalism resulting from the Gauss approximation, as well as the basic propagation matrices, are presented in the first section of this paper. There exist certain planes within an optical system that play an important role. This is the case for the couples of conjugate planes, because the intensity distribution across one plane is an image of the intensity distribution across the other plane. The second section is devoted to the establishment of the transfer matrix between two conjugate planes of an optical system. The notion of power introduced at the end of this section is coherent with the refractive power of a spherical interface. The intrinsic characteristics and properties of a focal optical system within the Gauss approximation are contained in the cardinal elements. With the power, they provide the necessary information for determining, within the frame of paraxial optics, the location and the size of the image of an object given by the optical system. These cardinal elements are described in the third section. When studying optical systems within the Gauss approximation, it is sometimes necessary to determine the location and the size of the image of an object. This can be done in a geometrical manner by considering particular rays, but the transfer matrix of the optical system provides an elegant and accurate way to reach this goal. Another algebraic way is given by the cardinal elements themselves. These two approaches are compared in the fourth section. The Newton and the Descartes relations suffer from a restricting assumption: the optical system should be a focal one, otherwise the cardinal elements would not be defined. This is not the case of the homographic relation which is still valid for nonfocal optical systems.

When studying complex optical systems, it is necessary to determine the path of the light with a greater accuracy than that obtained in the paraxial approximation. This may be done with the aid of elementary geometry, by successive application of

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the Snell-Descartes laws of refraction (or reflection). This method, which is known as ray tracing, is extensively used in the practical study of complex optical instruments. Since in an ideal system all rays that form an image are concurrent at the same image point, only two rays need to be traced to determine the image point at their intersection point. However, as we will see in the two last sections, because of geometrical aberrations, marginal rays are not concurrente at a single point, while paraxial ones are. This is why ray tracing is the only way to properly take into account aberrations in an optical system. The very last section of this work is concerned by the comparison between the Gauss approximation and the ray tracing of marginal rays, in the study of the correction to be done to a human eye suffering from ametropia. As expected, the ray tracing approach gives a significant improvement and could be very helpful for solving the problems encountered when considering this pathology.

2. GAUSS APPROXIMATION

The Gauss approximation is the linear approximation of geometrical optics. Within this frame, the properties of rays travelling through an optical system can be treated with an elegant matrix formalism.[1] We first define the column vector $\underline{X}$ whose complex components are the spatial coordinates $\underline{x} = x + iy$ in a front plane perpendicular to the optical axis z, and the optical angles $n\underline{\alpha} = n(\alpha + i\beta)$ of a point lying in a medium with refractive index n:

$$\underline{X} = \begin{pmatrix} \underline{x} \\ n\underline{\alpha} \end{pmatrix} \quad \text{with} \quad \underline{x} = x + iy \quad \text{and} \quad \underline{\alpha} = \alpha + i\beta. \tag{1}$$

For reasons that will be explained later in this section, it is preferable to use optical angles rather than just geometrical ones.

2.1. Propagation through free space

Geometrical rays travel in straight lines in a medium with a constant refractive index n. Therefore the effect of propagation through free space is simply to translate the location of the ray, in proportion to the angle at which it travels, and to leave the angle of the ray with the optical axis unchanged. Consequently, the transfer matrix describing the propagation between two front planes A_1xy and A_2xy (see Fig. 1) is given by:

$$\underline{X}_2 = \mathcal{T}(\overline{A_1A_2})\underline{X}_1 \quad \text{with} \quad \mathcal{T}(\overline{A_1A_2}) = \begin{pmatrix} 1 & \overline{A_1A_2}/n \\ 0 & 1 \end{pmatrix}. \tag{2}$$

It is worthy of note that the effective length of propagation involved is not the algebraic drift distance $\overline{A_1A_2}$, but the reduced length $\overline{A_1A_2}/n$ which takes into account the refractive index of the medium.

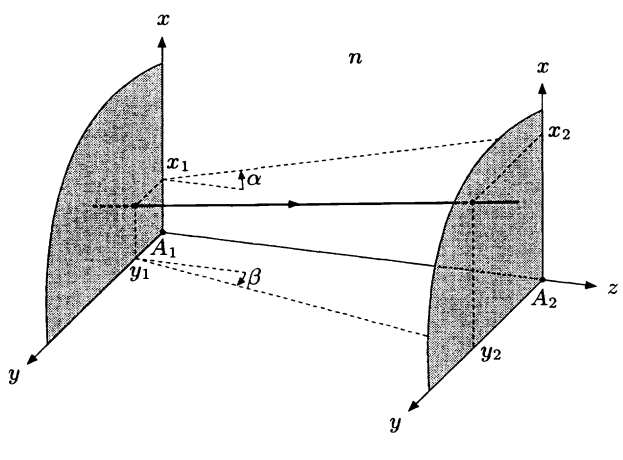


Figure 1. ***Propagation through free space.*** The ray travels in an homogeneous medium with constante refractive index n. It leaves the first front plane from $\underline{x}_1 = x_1 + iy_1$ with an optical angle $n\underline{\alpha}_1 = n(\alpha + i\beta)$, and hits the second one with the same optical angle $n\underline{\alpha}_2 = n(\alpha + i\beta)$ in $\underline{x}_2 = x_2 + iy_2$.

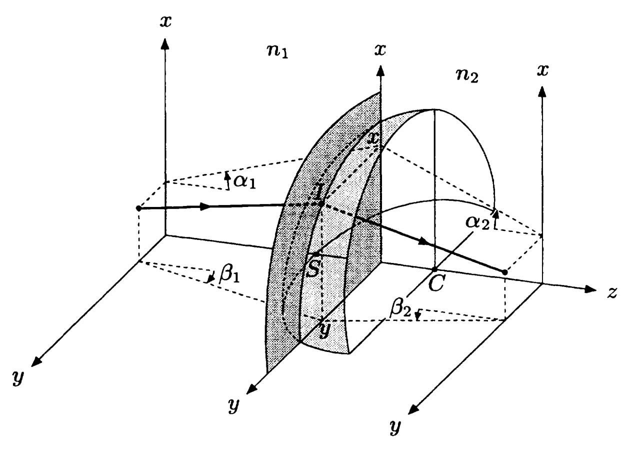


Figure 2. ***Refraction at a spherical interface.*** The ray travels through a spherical interface with extreme refractive index n_1 and n_2. It hits the refractive surface in $\underline{x}_1 = x + iy$ with an optical angle $n_1\underline{\alpha}_1 = n_1(\alpha_1 + i\beta_1)$, and leaves it from the same point $\underline{x}_2 = x + iy$ with an optical angle $n_2\underline{\alpha}_2 = n_2(\alpha_2 + i\beta_2)$.

2.2. Refraction (or reflection) at a spherical interface

At a spherical interface between an initial medium with refractive index n_1 and a final one with refractive index n_2, the position of a geometrical ray is not changed, but the optical angle varies according to the Snell-Descartes laws. In the linear approximation of geometrical optics, the transfer matrix for a spherical interface with algebraic radius $\overline{R} = \overline{SC}$, where S is the vertex of the spherical surface and C is the centre of curvature (see Fig. 2), is given by:

$$\underline{X}_2 = \mathcal{R}(S)\underline{X}_1 \text{ with } \mathcal{R}(S) = \begin{pmatrix} 1 & 0 \\ -V & 1 \end{pmatrix}, \text{ where } V = \frac{n_2 - n_1}{\overline{R}}. \tag{3}$$

Note that the matrix governing the reflection on a spherical mirror lying in an homogeneous medium with refractive index n is analogous to the matrix $\mathcal{R}(S)$ with $V = -2n/\overline{R}$. In both cases, a positive value for $\overline{R}$ means a convex surface encountered from left to right, while a negative value signifies a concave surface. The effect of the refraction (or reflection) on the propagation is governed by the sign of the refractive (or reflective) power V: a positive value signifies a convergent refractive (or reflective) interface, while a negative value signifies a divergent interface.

Remark. Because we have chosen to use optical angles rather than geometrical ones in (1), the determinant of the elementary matrices (2) and (3) is equal to unity, and the bottom-left element of the transfer matrix (3) for a spherical interface is equal to the opposite value of the power V. Moreover, according to (3), the transfer matrix for a planar interface ($\overline{R} \to \infty$) between a medium of refractive index n_1 and a medium of refractive index n_2 is reduced to the identity matrix.

2.3. Propagation through a centered optical system

Consider now a complex optical system consisting of regions of free space with a constant refractive index separated by spherical refracting surfaces (see Fig. 3): between the input and the output front planes, Exy and Sxy, the optical system is homogeneous step by step. Propagation through this system can be treated with the elementary matrices (2) and (3). Denoting by $\underline{X}_e$ and $\underline{X}_s$ the input and output vectors, we therefore have:

$$\underline{X}_s = \mathcal{T}(\overline{S_pS})\mathcal{R}(S_p)\cdots\mathcal{T}(\overline{S_1S_2})\mathcal{R}(S_1)\mathcal{T}(\overline{ES_1})\underline{X}_e. \tag{4}$$

The product of these elementary matrices, written from right to left following the path of the light, is the transfer matrix of the centered optical system within the Gauss approximation:

$$T(\overline{ES}) = \mathcal{T}(\overline{S_pS})\mathcal{R}(S_p)\cdots\mathcal{T}(\overline{S_1S_2})\mathcal{R}(S_1)\mathcal{T}(\overline{ES_1}). \tag{5}$$

It can be written in the shorter form:

$$T(\overline{ES}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6}$$

All the elementary matrices $\mathcal{T}$ and $\mathcal{R}$ involved in the product (5) have their determinant equal to unity. Consequently, according to a basic property of linear algebra, the determinant of $T(\overline{ES})$ is also equal to unity.

Remark. When studying catadioptric systems like optical cavities, it is useful to express the transfer matrix describing propagation from S to E. It turns out that $T(\overline{SE})$ can be derived from $T(\overline{ES})$ by simply permuting the diagonal elements a and d.

3. TRANSFER MATRIX

The typical kind of ray propagation problem, that must be solved in order to study the properties of an optical system, is shown on Fig. 3. The goal is to determine the position $\underline{x}_2$ and the angle $\underline{\alpha}_2$ of the output ray in a front plane A_2xy, for every possible $\underline{x}_1$ and $\underline{\alpha}_1$ associated with an input ray crossing a front plane A_1xy, after a first free propagation in a medium with refractive index n_o, travelling through an optical system, and a last free propagation in a medium with refractive index n_i.

3.1. Transfer matrix between two front planes

According to equations (2) and (3), the transfer matrix between two front planes A_1xy (in front of the input plane Exy) in the object space, with refractive index n_o, and A_2xy (beyond the output plane Sxy) in the image space, with refractive index n_i (see Fig. 3), is given by the product $\mathcal{T}(\overline{SA_2})T(\overline{ES})\mathcal{T}(\overline{A_1S}) = T(\overline{A_1A_2})$. Setting $z_1 \equiv \overline{EA_1}$ and $z_2 \equiv \overline{SA_2}$, and denoting by $T_{ij}(A)$ the elements of the transfer matrix for the two points A_1 and A_2, we therefore have:

$$\begin{pmatrix} T_{11}(A) & T_{12}(A) \\ T_{21}(A) & T_{22}(A) \end{pmatrix} = \begin{pmatrix} 1 & z_2/n_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -z_1/n_o \\ 0 & 1 \end{pmatrix}, \tag{7}$$

that is to say:

$$\begin{aligned} T_{11}(A) &= a + c\frac{z_2}{n_i} \qquad & T_{12}(A) &= -a\frac{z_1}{n_o} + b + \frac{z_2}{n_i}(-c\frac{z_1}{n_o} + d) \\ T_{21}(A) &= c & T_{22}(A) &= d - c\frac{z_1}{n_o} \end{aligned} \tag{8}$$

The only element of the four which is independent from the two points A_1 and A_2 is $T_{21}(A) = c$. Thus, this coefficient is an intrinsic characteristic of the optical system. By definition, the power of a focal optical system is the opposite value of this element:

$$V \equiv -c. \tag{9}$$

This definition is coherent with the power of a spherical interface (3). A positive value for V signifies a convergent optical system, while a negative one signifies a divergent system. If $V = 0$, the system is said to be nonfocal.

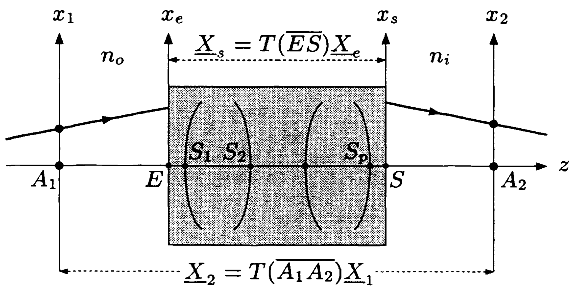


Figure 3. ***Propagation through an optical system.***

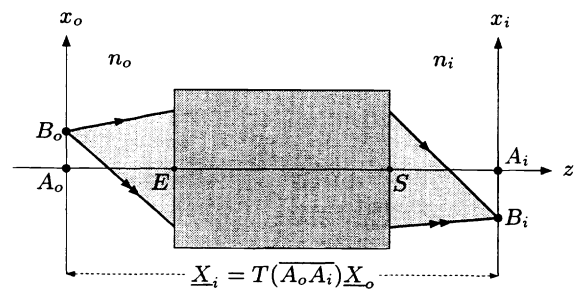


Figure 4. ***Transfer between two conjugate planes.***

3.2. Transfer matrix between two conjugate planes

Suppose now the two planes A_1xy and A_2xy are two conjugate planes A_oxy and A_ixy (see Fig. 4). Substituting $z_o \equiv \overline{EA_o}$ and $z_i \equiv \overline{SA_i}$ for z_1 and z_2 in (7), equations (8) become:

$$\begin{aligned} T_{11}(A) &= a + c\frac{z_i}{n_i} \qquad & T_{12}(A) &= -a\frac{z_o}{n_o} + b + \frac{z_i}{n_i}(-c\frac{z_o}{n_o} + d) \\ T_{21}(A) &= c & T_{22}(A) &= d - c\frac{z_o}{n_o} \end{aligned} \tag{10}$$

Due to the fact that the two off-axis points B_o and B_i are conjugate ones, the spatial coordinates $\underline{x}_i$ of B_i are independent of the angle $\underline{\alpha}_o$. As a consequence, it turns out that $T_{12}(A) = 0$. Hence, $T_{11}(A) = \underline{x}_i/\underline{x}_o \equiv G_t$, where G_t is the transversal magnification, and $T_{22}(A) = (n_i\underline{\alpha}_i/n_o\underline{\alpha}_o)_{\underline{x}_o=0} \equiv G_a n_i/n_o$, where G_a is the angular magnification. Finally, the transfer matrix between two conjugate planes is given by:

$$T(\overline{A_oA_i}) = \begin{pmatrix} G_t & 0 \\ -V & (n_i/n_o)G_a \end{pmatrix}. \tag{11}$$

Since the determinant of $T(\overline{A_oA_i})$ is equal to 1, we thus have:

$$\frac{n_i}{n_o}G_tG_a = 1 \text{ with } G_t \equiv \underline{x}_i/\underline{x}_o \text{ and } G_a \equiv (\underline{\alpha}_i/\underline{\alpha}_o)_{\underline{x}_o=0}. \tag{12}$$

This relation is the linear approximation of the Abbe relation: it is known as the Lagrange-Helmoltz relation.

4. CARDINAL ELEMENTS

We consider in this section an optical system of refracting surfaces characterized by the transfer matrix given by (6), with $V \neq 0$. There exist certain points and planes within such a system that play an important role. These intrinsic characteristics are known as the cardinal elements. With the power of the optical system, they contain the necessary information for determining, within the Gauss approximation, the location and the size of the image of an object given by the centered optical system. We first recall the definition of the focal lengths, then we give the definition of these cardinal elements: principal (or unit) planes, nodal points and focal planes.

4.1. Focal lengths

By definition, the image and object focal lengths, f_i and f_o, of an optical system with power V lying between an initial medium with refractive index n_o and a final medium with refractive index n_i, are the signed quantities:

$$f_i \equiv \frac{n_i}{V} \text{ and } f_o \equiv -\frac{n_o}{V}. \tag{13}$$

According to the meaning of the sign of the power, $f_i > 0$ and $f_o < 0$ for a convergent optical system, while for a divergent one $f_i < 0$ and $f_o > 0$. If $V = 0$, when the system is nonfocal, the focal lengths and the other cardinal elements are not defined.

4.2. Principal planes

These front planes are conjugate to one another with a transversal magnification G_t between them equal to unity (this is why these planes are often called unit planes). As a consequence, the transfer matrix for the passage between the object and the image principal planes, H_oxy and H_ixy (see Fig. 5), is given by:

$$T(\overline{H_oH_i}) = \begin{pmatrix} 1 & 0 \\ -V & 1 \end{pmatrix}. \tag{14}$$

Comparing the elements of this matrix with those of the transfer matrix of the system between Exy and Sxy (6), it turns out that we therefore have:

$$\overline{SH_i} = f_i(a-1) \text{ and } \overline{EH_o} = f_o(d-1). \tag{15}$$

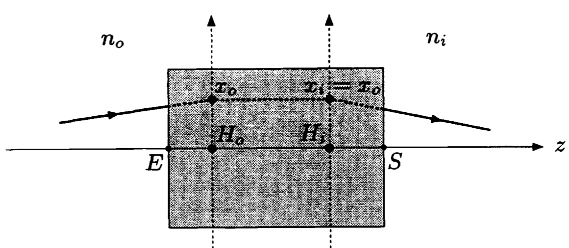


Figure 5. ***Principal planes:*** $G_t = 1$.

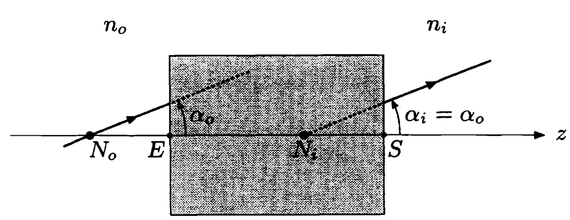


Figure 6. ***Nodal points:*** $G_a = 1$.

4.3. Nodal points

These two points, N_o and N_i, are conjugate points on the optical axis such that the angular magnification G_a between them is equal to unity: any incident ray crossing N_o leaves the system from N_i with the same direction as the incident one. Consequently, the transfer matrix between the object and the image nodal points (Fig. 6), is given by:

$$T(\overline{N_oN_i}) = \begin{pmatrix} n_o/n_i & 0 \\ -V & n_i/n_o \end{pmatrix}. \tag{16}$$

Comparing again the elements of this matrix with those of the transfer matrix of the system (6), their position with respect to E and S is given by the relations:

$$\overline{SN_i} = f_i(a - \frac{n_o}{n_i}) \text{ and } \overline{EN_o} = f_o(d - \frac{n_i}{n_o}). \tag{17}$$

It is sometimes convenient to evaluate the position of these points with respect to the principal planes. It is easy to show that we then have: $\overline{H_iN_i} = \overline{H_oN_o} = f_i + f_o$. According to the definition of the focal lengths (13), if the extreme mediums are identical ($n_o = n_i$), we therefore have $\overline{H_iN_i} = \overline{H_oN_o} = 0$: the nodal points coincide with the principal points.

4.4. Focal planes

Despite the notation, F_o and F_i, the object and the image focal points are not a pair of conjugate points on the optical axis. The mapping from F_o to F_i is one that maps angles into positions, and positions into angles:

$$T(\overline{F_oF_i}) = \begin{pmatrix} 0 & 1/V \\ -V & 0 \end{pmatrix}. \tag{18}$$

Coming back to the relation $\underline{X}_s = T(\overline{ES})\underline{X}_e$, any incident ray coming from F_o emerges parallel after travelling through the optical system: $\underline{\alpha}_s = 0$ whatever $\underline{x}_e$ and $\underline{\alpha}_e$ (Fig. 7). Likewise, F_i is the point of convergence of any incident ray parallel to the optical axis: $\underline{\alpha}_e = 0$ whatever $\underline{x}_s$ and $\underline{\alpha}_e$ (Fig. 7). Thus, according to (6) and (13), the location of F_o and F_i with respect to E and S is given by the relations:

$$\overline{SF_i} = f_i a \quad \text{and} \quad \overline{EF_o} = f_o d. \tag{19}$$

According to equations (15) and (19), the algebraic distances between the principal planes and the focal points are nothing but the focal lengths: $\overline{H_iF_i} = \overline{H_iS} + \overline{SF_i} = f_i$ and $\overline{H_oF_o} = \overline{H_oE} + \overline{EF_o} = f_o$.

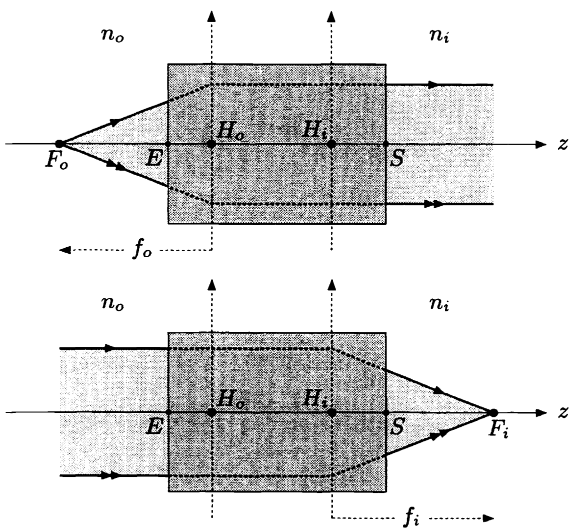


Figure 7. *Focal points and focal lengths.*

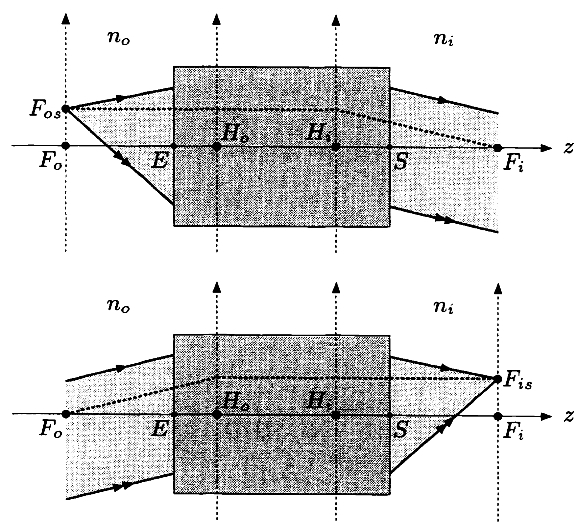


Figure 8. *Focal planes and secondary focal points.*

The object and the image focal planes F_oxy and F_ixy are the planes perpendicular to the optical axis erected through the focal points F_o and F_i. It is well-known that F_ixy is the conjugate of the object plane lying at infinity. Likewise, F_oxy is the conjugate of the image plane lying at infinity. The points lying in these planes, except F_o and F_i, are called secondary focal points (see Fig. 8).

4.5. Graphical determination of an image point

The properties of the cardinal elements can be used for the graphical determination of the image point B_i of an off-axis object point B_o. From a geometrical point of view, two rays entering the system from B_o are necessary to determine B_i at the intersection of the two emerging rays.[2] However, it is recommended to use three particular rays[1]:

i) the ray entering the system from B_o parallel to the optical axis,
ii) the ray entering the system from B_o and crossing the object focal point F_o,
iii) the ray entering the system from B_o and crossing the object nodal point N_o.

5. HOMOGRAPHIC RELATION

When studying optical systems, within the Gauss approximation, the game is sometimes to determine the location and the size of the image of an object given by the system. This can be done in a geometrical manner by considering particular rays (as explain above in the previous section), but the knowledge of the transfer matrix of the optical system provides an elegant and accurate way to reach this goal. Another algebraic way is given by the cardinal elements themselves.

Coming back to the transfer matrix between two conjugate planes (11) and to equations (10) with $T_{12}(A) = 0$, we thus have:

$$\frac{z_i}{n_i} = \frac{a\dfrac{z_o}{n_o} - b}{V\dfrac{z_o}{n_o} + d} \quad \text{with } z_o \equiv \overline{EA_o} \text{ and } z_i \equiv \overline{SA_i}. \tag{20}$$

This relation is known as the homographic relation. Provided that the transfer matrix of the optical system $T(ES)$ is perfectly known, it is thus possible to compute the location $z_i \equiv \overline{SA_i}$ of the image of an object given by the system from the location of the object $z_o \equiv \overline{EA_o}$, even if the optical system is a nonfocal one ($V = 0$). The transfer matrix between the two conjugates points A_o and A_i thus obtained can be easily derived from the transfer matrix of the optical system: $T(\overline{A_oA_i}) = \mathcal{T}(z_i)T(\overline{ES})\mathcal{T}(-z_o)$. According to (11) and (12), the transversal magnification G_t is equal to the upper-left element of $T(\overline{A_oA_i})$, and the angular magnification G_a is related to its lower-right element. It is thus possible to compute the components of the vector $\underline{X}_i$ from those of $\underline{X}_o$: $\underline{x}_i = G_t\underline{x}_o$ and $\underline{\alpha}_i = G_a\underline{\alpha}_o$.

5.1. Descartes relation

Measuring the location of A_o and A_i with respect to the principal points H_o and H_i, relation (20) becomes:

$$\frac{f_i}{p_i} + \frac{f_o}{p_o} = 1 \text{ where } p_o \equiv \overline{H_oA_o} \text{ and } p_i \equiv \overline{H_iA_i}. \tag{21}$$

This relation is known as the Descartes relation with regards to the principal points. The transversal and the angular magnifications are given by:

$$G_t = \frac{n_o\, p_i}{n_i\, p_o} \text{ and } G_a = \frac{p_o}{p_i}. \tag{22}$$

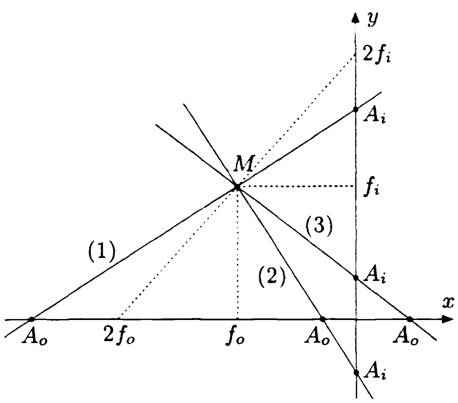


Figure 9. ***Geometrical interpretation of the Descartes relation.*** When $p_o \equiv \overline{H_oA_o}$ varies from $-\infty$ to $+\infty$, $p_i \equiv \overline{H_iA_i}$ is obtained by extending the straight line A_oM to its intersection with the y-axis.
For a convergent system ($f_i > 0$ and $f_o < 0$), M lies in the upper-left quadrant. Three cases should be considered:
(1) the object A_o is in front of F_o, the image A_i is behind F_i,
(2) the object A_o is in between F_o and H_o, the image A_i is in front of H_i,
(3) the object A_o is behind H_o, the image A_i is in between F_i and H_i.
Note that, for an object in front of H_o at a distance equal to $2f_o$, the image A_i is behind H_i at a distance $2f_i$.
For a divergent system ($f_i < 0$ and $f_o > 0$), three cases should be also considered, now in the lower-right quadrant

5.2. Newton relation

Likewise, measuring the location of A_o and A_i with respect to the focal points F_o and F_i, relation (20) becomes:

$$\sigma_i\sigma_o = f_if_o \text{ where } \sigma_o \equiv \overline{F_oA_o} \text{ and } \sigma_i \equiv \overline{F_iA_i}. \tag{23}$$

This relation is known as the Newton relation with regards to the focal points. The transversal and the angular magnifications are now given by:

$$G_t = -\frac{\sigma_i}{f_i} = -\frac{f_o}{\sigma_o} \text{ and } G_a = \frac{\sigma_o}{f_i} = \frac{f_o}{\sigma_i}. \tag{24}$$

Remark. Relations (21) and (23) provide two algebraic ways to compute the location of the image of an object given by the system from the location of that object. In this sense they are as useful as the homographic relation. However, they suffer from a restricting assumption: the optical system should be a focal one ($V \neq 0$), otherwise the cardinal elements would not be defined. This is not the case of (20) which is still valid for nonfocal optical systems ($V = 0$).

5.3. Geometrical illustration

A nice graphical illustration of the Descartes relation has been given by H. Bouasse[3] in 1947 (see Fig. 9). Indeed, this relation can be seen as the locus of a point $M(f_o, f_i)$ belonging to the straight line $x/p_o + y/p_i = 1$ connecting $A_o(p_o, 0)$ to $A_i(0, p_i)$. This construction is interesting above all for discuting the different cases of the respective positions of an object and its image.

6. RAY TRACING

When studying complex optical systems it is necessary to determine the path of the light with a greater accuracy than that given by the Gauss approximation. This may be done with the aid of elementary geometry, by successive application of the Snell-Descartes laws of refraction (or reflection). This method, which is known as ray tracing, is extensively used in the practical study of complex optical instruments.

6.1. Numerical ray tracing

Ray tracing falls within the realm of geometric optics: light travels in straight lines which are only deviated by reflections or refractions due to a change in refractive index, and can be traced using conventional geometry and trigonometry. Ray tracing can be decomposed in four steps: constructing an appropriate straight line to represent the ray, locating the point of intersection with the next interface, refracting the ray by applying the Snell-Descartes laws, and representing the refracted ray by another straight line. In a rotationally symmetric optical system, meridional rays stay in the same plane as they are refracted, while skew rays does not. Thus, ray tracing meridional rays is a two dimensional exercise, while ray tracing of skew rays is a three dimensional one. Consequently, for practical reasons, only meridional rays are considered in this section, but despite the complexity, there is no difficulty to extend this method to skew rays.

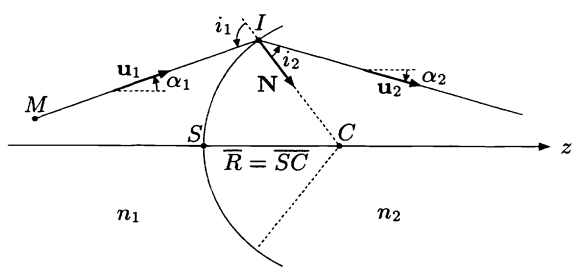


Figure 10. ***Refraction at a spherical interface with sign conventions.*** The common cartesian and trigonometric sign conventions are used with the axis origin placed anywhere on the optical axis. The incident ray travelling from M is characterized by the unit vector $\mathbf{u}_1$, while the refracted one is represented by the unit vector $\mathbf{u}_2$ and the intersection point I on the refractive surface. The unit vector $\mathbf{N}$, normal to the surface at I, is always oriented towards the center C of the spherical surface. These vectors are related together by the Snell-Descartes law whose vector writing is $n_2\mathbf{u}_2 - n_1\mathbf{u}_1 = a\mathbf{N}$, where a is a real number.

Provided that the sign convention is that shown on the previous figure, tracing the refracted ray $(I, \mathbf{u}_2)$ from the incident one $(M, \mathbf{u}_1)$ and from the characteristics of the interface $(n_1, n_2, \overline{R})$ does not rise any difficulty and could be achieve in three numerical steps:

step 1: Calculation of the intersection point I.
Calculation of the surface normal $\mathbf{N}$ at I.
step 2: Calculation of the angle of incidence at I: $i_1 = \arccos(\mathbf{N} \cdot \mathbf{u}_1)$.
Calculation of the angle of refraction at I: $i_2 = \arcsin(n_1 \sin i_1 / n_2)$.
step 3: Calculation of the refracted ray: $\mathbf{u}_2 = (n_1\mathbf{u}_1 + a\mathbf{N})/n_2$, with $a = n_2 \cos i_2 - n_1 \cos i_1$.

6.2. Computer aided ray tracing

The above procedure can readily be applied to ray tracing through an optical system of any number of interfaces. After refraction, the ray is transferred to the next interface, where the next refraction takes place. At this interface, the new angle α_1 is the old one α_2, and the new starting point is the previous intersection point I. The procedure is performed until the ray is stopped by a diaphragm in the system, or leaves it through the exit pupil. In this latter case, it is straightforward to compute the intersection of the exit ray with the optical axis or with any front plane perpendicular to the optical axis.

Thanks to Object Oriented Programing (OOP) and to Rapid Application Development (RAD) tools, the implementation of this method with a high level language does not raise any difficulty. The language adopted for this work was `C++`, and we are now working on a `Java` implementation.

7. APPLICATION TO THE EYE

Perhaps the simplest of the optical instruments is that consisting of a single convergent lens forming a real image of an object upon a light-sensitive surface. Examples of more complicated optical system are found in the photographic camera and in the eye. Inasmuch as the eye forms an integral part of many optical systems, an understanding of its characteristics is an essential part of instrumental optics. The present section will include a description of a few of its features, and will focus on the correction of spherical refractive errors.

7.1. Brief anatomy and functions of the human eye

The optical system of the eye is not a rotationally symmetric one. Thus the eye does not have a true optical axis. The visual axis does not coincide with the best fit optical axis: these axis are tilted to each other by about 5°. A cross-section of the human eye is shown in Figure 11, giving only the most relevant optical components.[4] The light rays enter the eye through and are refracted by the cornea, the front shiny transparent surface of the eye. Then they are further refracted by the lens, bringing them to a focus point on the retina, a layer of light-sensitive cells (the cones and the rods). The lens lies in two extreme mediums with different refractive index: on the anterior face is the aqueous, a transparent liquid; and on the posterior face is the vitreous, a gelatinous liquid.

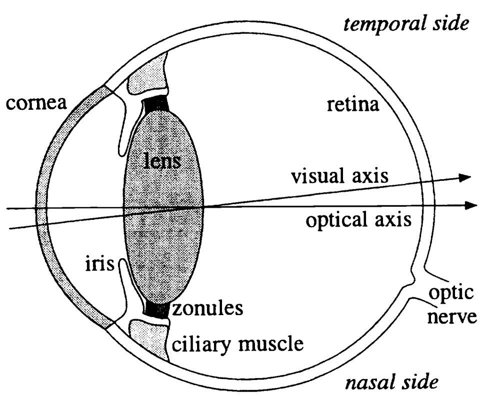


Figure 11. ***Cross-section of the human eye.*** Image forming light enters the eye through and is refracted by the cornea. It is further refracted by the lens, bringing it to a focus on the retina. Whereas the power of the cornea is constant, the power of the lens depends upon the level of accommodation. Accommodation is the process by which the refractive power of the eye changes to allow objects of interest at different distances to be sharply imaged on the retina. It is controlled by the ciliary muscle which can relax or contract, causing the zonules supporting the lens to be contracted or relaxed, thus changing the shape of the lens capsule. The diameter of the incoming beam of light is controlled by the iris, which is the aperture stop of the eye. The optical system of the eye is not a rotationally symmetric one. The visual axis does not coincide with the best fit optical axis: these axis are tilted to each other by about 5°.

Of the two refracting elements, the cornea has the greater refractive power (about two third of the total power). However, whereas the power of the cornea is constant, the radius of curvature of the lens capsule may be altered by muscular contraction, to serve the purpose of focussing. Thus the power of the lens depends upon this level of adjustment, or accommodation. For example, when the eye needs to focus on closer objects, the ciliary muscles contract, causing the suspensory ligaments (the zonules) supporting the lens to relax. This allows the lens to take a more spherical shape, and to change its refractive power accordingly. When the eye has to focus on more distant objects the reverse applies. There are physical limits to how far the ciliary muscles can relax and contract, and how far the lens can be stretched and contracted. Thus there are upper and lower limits to the refractive power of the eye, and in turn farthest and closest distances of vision (see Fig. 12). These extremes distances refer to the far point R_o (*remotum*) and to the near point P_o (*proximum*).

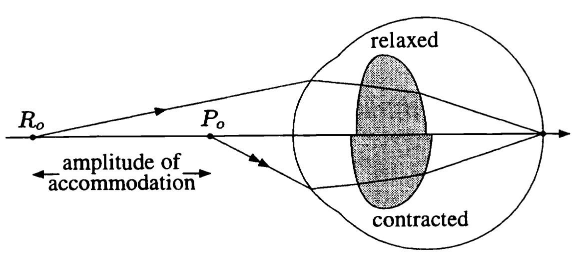


Figure 12. ***Accommodation range of the human eye.*** When the ciliary muscles are completely relaxed, the power of the lens is minimum and the eye is focussed on the far point R_o (distance vision). When they are maximally contracted, the lens has its greatest refractive power and the eye is focussed on the near point P_o (near vision). The difference between these extreme powers of the eye is called the amplitude of accommodation.

Because of large variations in the dimensions of the components of real eyes, it is not easy to define a standard eye. However, it is usually assumed that a normal eye should be focussed at infinity when the accommodation is relaxed: that is the normal eye has a far point at infinity. This eye is termed emmetropic, all the other ones are called ametropic and are regarded as having some kind of refractive errors. These refractive errors can be categorized as either spherical (myopia, hypermetropia and presbyopia) or cylindrical (astigmatism):

i) myopia (or short sightedness), in which the far point is at a finite distance in front of the eye. This is corrected by means of a divergent lens placed in front of the eye (see Fig. 13).

ii) hypermetropia (or long sightedness), in which the far point is behind the eye. Correction is obtained by means of a convergent lens (see Fig. 13).

iii) presbyopia is the refractive error of an eye with zero or very little amplitude of accommodation. It is due to advancing age, and it cannot be corrected by a single power lens. In practice, this defect is corrected by means of multifocal lenses.

iv) astigmatism, in which the power of the eye differs in different planes containing the optical axis. This defect is corrected by means of a suitable toric lens.

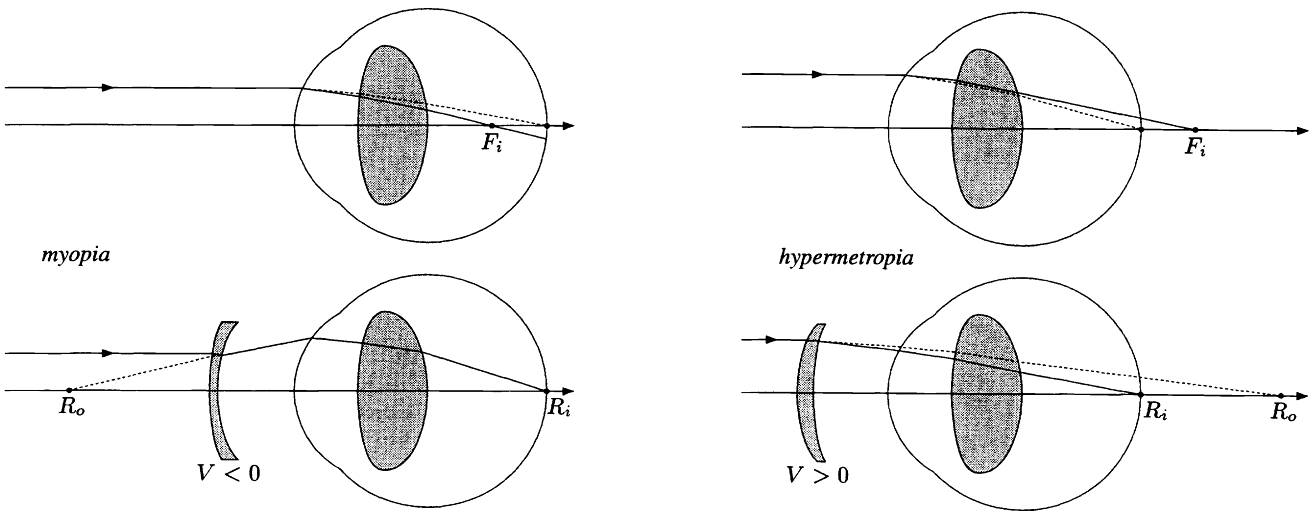


Figure 13. ***Spherical refractive errors and their correction.*** On the left hand, in an eye suffering from myopia the rays from an infinitely distant object point reach a focus in front of the retina. This refractive error arises because the power of the eye is too great for its axial length: it is corrected by means of a divergent lens placed in front of the eye. The negative power of the ophthalmic lens can be thought of as either compensating for the excess power of the eye, or instead imaging object at infinity onto the back focal plane of the lens, which is coincident with the far point plane of the myopic eye. On the right hand, in an eye suffering from hypermetropia the rays from an infinitely distant object point reach a focus behind the retina. This refractive error arises because the power of the eye is too low for its axial length: correction is obtained by means of a convergent lens. The positive power of the ophthalmic lens can be thought of as either compensating for the insufficient power of the eye, or instead imaging distant object onto the far point plane of the hyperopic eye.

The dimensions of the eye and the characteristics of its optical components vary greatly from person to person, and some further depend upon accommodation level, age and certain pathological conditions. Despite these variations, average values have been used to construct representative or schematic eyes. The standard model used for this work is the Le Grand model of a relaxed eye.[4] This model is a four interfaces model: the two first interfaces correspond to the cornea, the two last ones constitute the lens. The characteristics of each interface are given in the following table.

Surface	Distance (mm)	Curvature (mm)	Refractive index	Medium
			1.0000	air
$E \equiv S_1$	0.000	7.800		
			1.3771	cornea
S_2	0.550	6.500		
			1.3374	aqueous
S_3	3.600	10.200		
			1.4200	lens
$S \equiv S_4$	7.600	-6.000		
			1.3360	vitreous

Table 1. ***Le Grand theoretical relaxed eye.*** The transfer matrix between the anterior face of the cornea and the posterior face of the lens is approximatively equal to:

$$T(\overline{ES}) \simeq \begin{pmatrix} 0.74461 & 0.00545 \\ -59.94043 & 0.90442 \end{pmatrix}.$$

In addition to these values, it is important to note the position of the input pupil (the image of the iris) in the aqueous: 3.038 mm behind E.

The cardinal elements can be easily derived from the transfer matrix $T(\overline{ES})$. For example, the eye length, given by the position of the image focal point on the retina, is equal to 24.197 mm. Likewise, the total power of the eye is equal to 59.940 δ, divided into 42.356 δ for the cornea (48.356 δ for the front face, and -6.108 δ for the back face) and 21.779 δ for the lens (8.098 δ for the front face, and 14.000 δ for the back face).

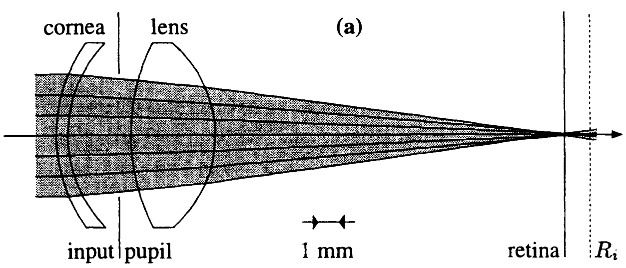
7.2. Correction of ametropias

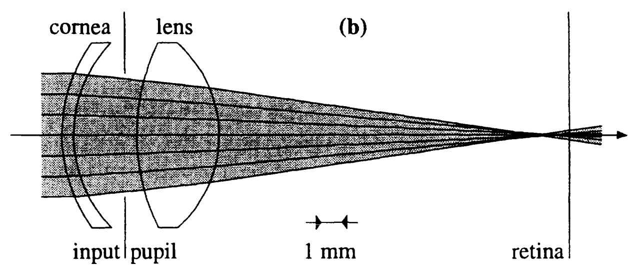
Nowadays, some ametropias of the eye (myopia and hypermetropia) could be corrected by means of a modification of the shape of the anterior face of the cornea.[5] A photo-ablation of a small piece of the cornea is obtained with the aid of an excimere laser. After cicatrization, the radius of curvature of the anterior face of the cornea is modified. The refractive power of the cornea is changed accordingly, and the ametropic eye becomes emmetropic. One of the problems accountered in practice is to evaluate the amount of cornea removal in order to give to the cornea the expected refractive power. We show in this section that a ray tracing approach could be very helpful to achieve this goal.

In clinical practice, the position of the far point is never measured directly, and alternative techniques are used to measure the myopia.[4] One method uses a number of lenses of different power at some well-known distance h (typical distances vary from about 12 mm to 15 mm) in front of the entrance vertex E of the cornea. Denoting by V_h the power of the lens which gives clear viewing for a target at infinity, and referring back to Fig. 13 within the Gauss approximation, the position of the far point is given by $z_o = \overline{ER_o} = 1/V_h - h$.

Within the framework of the Gauss approximation, the position of the retina with respect to the vertex of the posterior face of the lens, $z_i = \overline{SR_i}$, could be obtained with the aid of the homographic relation (20). Computing the radius of curvature of the anterior face of the cornea required to bring the image focus point on the retina does not raise any difficulty. Indeed this linear problem is solved with the aid of the matrix method by computing the transfer matrix (6), the refractive power (9), the focal length (13), and equating $\overline{SF_i}$ with $\overline{SR_i}$.

Another way to proceed is to determine the path of the light with a greater accuracy than that obtained in the paraxial approximation by using a numerical ray tracing procedure. The position of the retina is now defined as that of the plane perpendicular to the optical axis where the rays coming from the far point give the smallest spot. It is then easy to remove, with the aid of the computer, some piece of the cornea step by step (say by micrometrical slices) until rays coming from infinity converge to the retina. This procedure is illustrated on the following figures where the diameter of the input pupil has been set to 6 mm (which is an intermediate value between day and night vision).





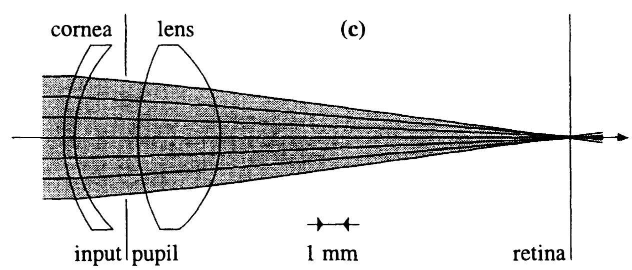


Figure 14. ***Correction of myopia with a ray tracing approach.***
(a) Location of the retina: rays coming from R_o, which is 265 mm in front of the cornea, converge 1.23 mm before the conjugate point R_i given by (20) in the Gauss approximation (dashed line).
(b) Before correction, rays coming parallel to the optical axis from infinity converge 1.27 mm in front of the retina (diameter of spot on the retina is 0.46 mm).
(c) After axial ablation of 31 μm, rays coming parallel to the optical axis from infinity converge on the retina (spot size on the retina less than 0.09 mm).

Both methods are compared on the following figure, where the amount of cornea removal has been computed for eyes suffering from myopia (in the range -12 δ to -3 δ) and for two sizes of the input pupil: 4 mm and 6 mm pupil diameter. It is not surprising to observe a difference between the two methods, since for severe myopia, that is when the far point is close to the anterior face of the cornea, the rays become marginal and do not converge at a single point on the optical axis. As a consequence, for severe myopia the paraxial approximation suggests to remove up to 10 μm more cornea than the marginal ray tracing approach. To understand the consequences of this difference, it is not pointless to keep in mind that, from a clinical point of view, the maximum amount of cornea that can be reasonably removed is about 100 μm out of 550 μm axial thickness (see Tab. 1).

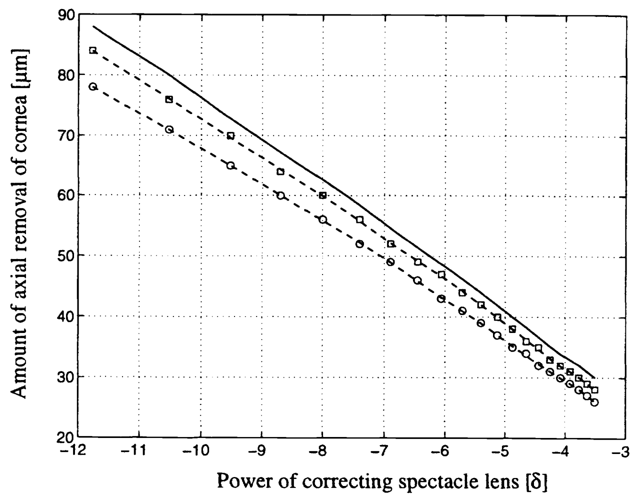


Figure 15. ***Correction of myopia.*** The range of myopia corresponds to far point distances between 100 mm and 300 mm in front of the anterior face of the cornea. Solid line is relative to the Gauss approximation, dashed lines to the ray tracing approach with two different diameters of the input pupil: 4 mm for the squares, 6 mm for the circles. The closest to the cornea is the far point and the larger is the input pupil, the more inclined on the optical axis are the rays, and thus the more different are the amounts of axial removal. For a given amount of cornea removal, the marginal ray tracing explains a myopia correction up to 2 δ stronger than the paraxial one. Moreover, for a given level of myopia, the Gauss approximation suggests to remove up to 10 % more cornea than the ray tracing approach, depending on the size of the input pupil.

The same approach could be easily applied to the correction of hypermetropia, where the amount of off-axis cornea removal has just an impact on the reduction of the radius of the anterior face of the cornea, in order to grow its refractive power. This is not the case for the myopia, because the axial removal increases the radius of curvature in order to reduce the refractive power of the eye, but also has an impact on the axial thickness of the cornea, and therefore on the position of the entrance vertex E.

8. CONCLUSION

We have shown the interest of the matrix method for studying complex optical systems within the frame of the Gauss approximation. Indeed, thanks to the introduction of the optical angle all the matrices involved in this approach have some nice properties. Moreover, it is the simplest way to obtain the cardinal elements, and it is well suited to the study of catadioptric systems with both refractive and reflective devices.
With regards to the determination of the location and size of the image of an object given by a centered system, the matrix method provides an elegant and accurate way with respect to geometrical one. However, the traditionnal relations of Descartes and Newton suffer from a restricting assumption which supposes the optical system to be a focal one. A more general approach is provided by the homographic relation, which is still valid for nonfocal systems.
From a pedagological point of view, the matrix method is thus a nice and precise method which does not exempt from paraxial ray tracing within the Gauss approximation. When it is necessary to determine the path of the light with a greater accuracy, the marginal ray tracing approach gives a significant improvement because it is the convenient way to take into account geometrical aberrations. Beyond the example chosen to illustrate the thought process, the two methods are necessary and complementary when studying complex optical systems.

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