Uniqueness condition for 1D signal reconstruction from power spectrum with related interference signal information

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ABSTRACT

Signal reconstruction from its power spectrum is an important technique widely useful in different application fields. However, uniqueness is generally hard to be guaranteed by usual approaches due to nonlinearity of the power spectrum measurement models. In this paper a very general structure of the 1-dimensonal signal with given power spectrum is established and on basis of this formulism some conditions are introduced to guarantee the uniqueness of solution in signal reconstruction. Such conditions are about the information of the interference signals and according to which an algorithm is established for signal construction. The algorithm is simple, efficient and its solution is unique in case of linear-phase signals. Some generalized and more practical conditions are also discussed.

Keywords: Power spectrum, signal reconstruction, non-linear measurement model, uniqueness, reference signal

1. INTRODUCTION

1.1 Basic problems and related works

Signal reconstruction problem widely appears in different application fields. In this problem the complete signal, i.e., both its magnitudes and phases at a finite number of specified time instants or spatial positions, need to be reconstructed from its power spectrum measurements[1,2].

Power spectrum measurement is essentially a non-linear observation model and only magnitude information of signal's Fourier transform is obtained[3]. As a result, uniqueness is generally hard to be guaranteed by usual approaches due to nonlinearity of the power spectrum measurement model and its information incompleteness. So far, most approaches are based on extending the sampling frequency, e.g., by doubling the number of sampling points, to provide sufficient measurements to guarantee uniqueness[4,5]. Such methods obvious have lots of limitations in some realistic applications, e.g., it is unfeasible to have adequate sampling density or speed with some acceptable cost, particularly for two or higher dimensional signals[6]. Even for one dimensional situation, e.g., the time sequence signal of finite length, simply enlarging its sample set is in many cases unfeasible in practice[7-9]. On the other hand, in many typical situations some additional information about the signal itself may be available which can be helpful for reconstructing the real signal to get the unique solution in some conditions. For example, in one dimensional photoelectric lattice and two-dimensional photoelectric membrane experiments not only the absorption spectrum (essentially the power spectrum of the anisotropic photoelectric material's permeability) can be measured but also the value of permeability itself can be measure at some special position[10]. In some nonlinear optical experiments, not only the field spectrum at given set of frequencies but also some additional information about the field itself or reference field at some special points can be available[11,12]. Naturally how to make use of such additional information to develop the reconstruction algorithm is a valuable problem which needs theoretical and practical investigation. However, so far there are only few works in this direction.

1.2 Contributions

In this paper a very general structure of the 1-dimensonal signal with given power spectrum is proved and on basis of this formulism some conditions are introduced to guarantee the uniqueness of solution in signal reconstruction. Such conditions are about some additional information of the interference signal which is useful and obtainable in the situation where the original spectrum as well as the interference spectrum of the signal with some finite-length reference signal is measured. On basis of this formulism and uniqueness condition, an algorithm is established for signal construction,

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which is simple, efficient and the solution is unique in case of linear-phase signals. Some generalized and more practical conditions are also discussed.

2. BASIC PROBLEM, MODELS AND ELEMENTARY PROPERTIES

Notations and Conventions All vectors $\boldsymbol{x}$, $\boldsymbol{y}$, etc. are column vectors, so $\boldsymbol{x}^T$, $\boldsymbol{y}^T$, etc. are raw vectors. $x(n)$ is the n-th component of vector $\boldsymbol{x}$. Z is the set of integers.

2.1 Basic model

The basic model for one-dimensional signal reconstruction is the following observation equation:

$$|\tilde{x}(\omega)|^2=|\sum_{n\in Z} x(n)\exp(-in\omega)|^2 \tag{1}$$

In this equation, $\boldsymbol{x}\equiv\{x(n)\}$ is a sequence of complex-number valued signals with N items, each item $x(n)=|x(n)|\exp[i\varphi(n)]$ where ω is the discretized frequency. The starting instant n_o and length N of such signal sequence is given, so without loss of generality, let $\{x(n)\}$ be $\{x(0), x(1), ..., x(N\text{-}1)\}$ where $x(0)$ and $x(\mathrm{N}\text{-}1)$ are non-zero while $x(n)=0$ for $n<0$ or $n\geq\mathrm{N}$. All unessential sampling parameters and scale factors are neglected.

The basic problem is: reconstruct signal $\boldsymbol{x}\equiv\{x(n)\}$ according to observations $\{|\tilde{x}(\omega_j)|\}$ which are a collection of data at discretized frequencies $\{\omega_j\}$.

It is emphasized that the observation $|\tilde{x}(\omega_j)|$ has only magnitude information in frequency domain, however, what is to be constructed is the complete signal $x(n)=|x(n)|\exp[i\varphi(n)]$ with information of both magnitude $|x(n)|$ and phase $\varphi(n)$.

In the following analysis the auto-correlation sequence plays an important role which definition is:

$$a(n)\equiv\sum_{k\in Z} \bar{x}(k)x(n+k) \tag{2}$$

The auto-correlation sequence $\boldsymbol{a}\equiv\{a(n)\}$ of signal $\{x(0), x(1), ..., x(N\text{-}1)\}$ is a sequence of size $2N\text{-}1$ where $\mathrm{a}(n)=0$ $|n|>N$. The Fourier spectrum of $\boldsymbol{a}$ and the Fourier spectrum of $\boldsymbol{x}$ are related by

$$\tilde{a}(\omega)\equiv\sum_{n,k\in Z} \bar{x}(k)x(n+k)\exp(-in\omega)=|\tilde{x}(\omega)|^2 \tag{3}$$

Note that the signal $\boldsymbol{x}$ of size N has $2N$ to-be-determined real-number valued variables: N magnitudes $|x(n)|$ and N phases $\varphi(n)$. As a result totally $2N$ observations $|\tilde{x}(\omega_j)|$ in model (1) are needed to reconstruct the complete signal $\boldsymbol{x}$. By the well-known interpolation formula, such observations can determine the continuous power spectrum $\tilde{a}(\omega)=|\tilde{x}(\omega)|^2$ of the auto-correlation sequence $\boldsymbol{a}$, i.e.,

$$\tilde{a}(\omega)=\sum_{j=1}^{2N} |\tilde{x}(\omega_j)|^2 \prod_{i=1,i\neq j}^{2N} \frac{\sin(\omega-\omega_i)/2}{\sin(\omega_j-\omega_i)/2} \tag{4}$$

where $\omega_i\text{-}\omega_j\neq 2m\pi$ for any $m\in\mathrm{Z}$. The auto-correlation sequence $\boldsymbol{a}$ can be completely determined by Fourier transform of (4). Due to non-linearity of the model, however, signal $\boldsymbol{x}$ itself cannot be completely determined by this approach. For example, it's easy to verify that all signals $\{\exp(-i\alpha)x(n)\}$, $\{x(n\text{-}m)\}$ and $\{\overline{x(-n)}\}$ have the same auto-correlation sequence as signal $\{x(n)\}$. Such signals are called hereafter the simple variant of the original signal $\{x(n)\}$, which have the same physical significance of $\boldsymbol{x}$ in practical applications. With this consideration, the signal reconstruction problem can be re-stated more exactly by:

Given observations $\{|\tilde{x}(\omega_j)|\}$ which are a collection of data at discretized frequencies $\{\omega_j\}$, reconstruct signal $\boldsymbol{x}\equiv\{x(n)\}$ which is unique in the sense that all different solutions are within the simple variant category of the exact signal $\boldsymbol{x}$.

As a result of the following analysis, the uniqueness cannot be simply reached by 2N observations, therefore some additional observations or constraints are necessary. Our analysis in the following also shows under which additional conditions the uniqueness can be really guaranteed.

2.2 Some related models

In practice some measurement models in different forms frequently appear, e.g., the electric conductivity measurement model used in the research of polymer electronic devices is:

$$|\tilde{x}(\omega)|^2=|\sum_{n\in Z} x(n)\mathrm{K}(\omega, n)|^2 \tag{5}$$

where the kernel function

$$K(\omega, n) \equiv (2\pi b)^{-1/2}\exp(-i\pi/4)\exp[(i/2b)(an^2-2\omega n+d\omega^2)]$$

By straightforward calculation, it holds that

$$|\tilde{x}(\omega)|^2=|\sum_{n\in Z} x(n)\exp[(ian^2/2b]\exp(-in\omega)|^2$$

i.e., model (5) can be reduced to model (1) if $x(n)\exp[(ian^2/2b]$ rather than $x(n)$ is regarded as the signal. In its application domain a and b are given parameters so signal reconstruction problem of model (5) is equivalent to the same problem of model (1).

Many other observation models in applications have the relations with model (1) similar as the above, hence model (1) is typical and the following analysis is focused on it.

3. GENERAL FORMALISM AND MAIN RESULT ON UNIQUENESS

Non-uniqueness is the most essential obstacle for signal reconstruction on basis of model (1). In this section the complete form of signal $\boldsymbol{x}$ is derived from measurement Equation (1). As a result, the most important impact factors for the uniqueness can be determined and furthermore the uniqueness condition can be established on basis of this analysis.

3.1 General formalism and main result

For a given sequence $\boldsymbol{a}\equiv\{a(n): n=1-N, \ldots, N-1\}$ of length N, define an associated polynomial of degree $2N$-2:

$$P_{\boldsymbol{a}}(z)\equiv\sum_{n=0}^{2N-2} a(n-N+1)z^n \tag{6}$$

For the auto-correlation sequence $\boldsymbol{a}$ of signal $\boldsymbol{x}$ it always holds that

$$\tilde{a}(\omega)=|\tilde{x}(\omega)|^2=\exp[-i(N-1)\omega)]P_{\boldsymbol{a}}(\exp(-i\omega)) \tag{7}$$

By Equation (2) we have $\overline{a(-n)}=a(n)$ so it's true that

$$\bar{z}^{2(N-1)}P_{\boldsymbol{a}}(1/\bar{z})=\overline{P_a(z)}$$

As a result, if γ is a non-zero root of $P_{\boldsymbol{a}}(z)$ then so is $1/\bar{\gamma}$, i.e., the roots of $P_{\boldsymbol{a}}(z)$ can be paired as $(\gamma, 1/\bar{\gamma})$ which are distributed inside and outside the unit circle symmetrically.

Equation (2) implies $\overline{a(1-N)}=a(N-1)=x(0)x(N-1)\neq 0$ (otherwise at least one of $x(0)$, $x(N-1)=0$ which conflicts the assumption for sequence $\boldsymbol{x}$), so $P_{\boldsymbol{a}}(0)\neq 0$. This implies:

$$P_{\boldsymbol{a}}(z)=a(N-1)\prod_{j=1}^{N-1}(z-1/\bar{\gamma}_j)(z-\gamma_j) \tag{8}$$

By (7) and $\tilde{a}(\omega)=|\tilde{x}(\omega)|^2$ there holds that

$$\tilde{a}(\omega)=|P_{\boldsymbol{a}}(\exp(-i\omega))|$$
$$=|a(N-1)||\prod_{j=1}^{N-1}|(\exp(-i\omega)-1/\bar{\gamma}_j)(\exp(-i\omega)-\gamma_j)|=|a(N-1)||\prod_{j=1}^{N-1}|\gamma_j|^{-1}\prod_{j=1}^{N-1}|\exp(-i\omega)-\gamma_j)|^2$$

For real-valued frequency ω then

$$\tilde{x}(\omega)=|a(N-1)|^{1/2}\prod_{j=1}^{N-1}|\beta_j|^{-1/2}\prod_{j=1}^{N-1}(\exp(-i\omega)-\beta_j) \tag{9}$$

where $\{\beta_j: j=1, \ldots, N-1$ and $\beta_j\neq 1/\bar{\beta}_k$ for any $j\neq k\}$ is a subset of roots.

Since signal $\boldsymbol{x}$ is determined by its spectrum $\tilde{x}(\omega)$, Equation (9) shows the general formalism of the solution to Equation (1). Nevertheless, it is not unique because different subset of $\{\beta_j\}$ of cardinality N-1 determines different solutions.

The above argument essentially proves the theorem which is the main result in this section:

Theorem 1 Any signal $\boldsymbol{x}$ implied by Equation (1), up to its simple variant, must has the spectrum in form of (9) where $\{\beta_j: j=1, \ldots, N-1$and $\beta_j\neq 1/\bar{\beta}_k$ for any $j\neq k\}$ is the subset of roots of polynomial $P_{\boldsymbol{a}}(z)$ defined in (6).

This theorem is the foundation to develop the signal reconstruction algorithm. An interesting corollary of this theorem is that if $P_{\boldsymbol{a}}(z)$ associated with the measurement data only has roots on the unit circle then the constructed signal is unique (up to simple variants). Such situation cannot occur in general, but the more is the number of roots on the unit circle on the complex plane, the less is the number of different solutions. Let L≡|{(γ_j, $1/\overline{\gamma_j}$): $\gamma_j \neq 1/\overline{\gamma_j}$} be the number of pairs not distributed along the unit circle, m_j be the multiplicity of the j-th pair (γ_j, $1/\overline{\gamma_j}$), then it is easy to verify that the number of signals implied by Equation (1) is at most:

$$(1/2)\prod_{j=1}^{L}(1+m_j)$$

In applications theorem 1 has another useful equivalent form which can be stated as the following.

Theorem 2 Let ***x*** and ***y*** be two signals of length N and both satisfying Equation (1), then there exist two signals ***u*** and ***v*** of finite lengths such that

$$\boldsymbol{x}=\boldsymbol{u}\ \#\ \boldsymbol{v} \quad \text{and} \quad \boldsymbol{y}=\{\exp[i\alpha]\overline{u(-n)}\}\#\{v(n-m)\}$$

where # is the convolution operator, a is some real-number constant and m is an integer constant.

The proof of this theorem is straightforward. Since by theorem 1 signal both ***x*** and ***y*** can be represented by (9) but with different set of roots {β_j: j=1, …, N-1} and {γ_j: j=1, …, N-1}, then

$$\tilde{x}(\omega)=\text{constant }\tilde{y}(\omega)\prod_{j=1}^{N-1}\frac{(\exp(-i\omega)-\gamma_j)}{(\exp(-i\omega)-\beta_j)}$$

Let {β_1, …, β_m} and {γ_1, …, γ_m} are the largest ono-intersection subset of roots where $|\gamma_j|\leq 1$, j=1, …, t and $|\gamma_j|>1$, j=t+1, …, m. Since the distribution of $P_a(z)$'s roots are symmetric relative to the unit circle, there should exist exactly t roots $\beta_j=1/\overline{\gamma_j}$, $j\leq$t and other m-t roots $\beta_j=1/\overline{\gamma_j}$, $j\geq$t+1, therefore:

$$\frac{\widetilde{x(\omega)}}{\widetilde{y(\omega)}}=\text{constant }\prod_{j=1}^{m}\frac{(\exp(-i\omega)-\gamma_j)}{(\exp(-i\omega)-1/\overline{\gamma_j})}=\text{constant }\exp[im\omega]\prod_{j=1}^{m}\frac{(\exp(-i\omega)-\gamma_j)}{(\exp(+i\omega)-\overline{\gamma_j})}=\text{constant }\exp[im\omega]\tilde{u}(\omega)/\bar{\tilde{u}}(\omega)$$

where $\tilde{u}(\omega)\equiv\prod_{j=1}^{m}(\exp(-i\omega)-\gamma_j)$. Equivalently:

$$\widetilde{x(\omega)}/\tilde{u}(\omega)=\text{constant }\exp[im\omega]\tilde{y}(\omega)/\bar{\tilde{u}}(\omega)$$

Let $\tilde{v}(\omega)\equiv\widetilde{x(\omega)}/\tilde{u}(\omega)$ then

$$\tilde{x}(\omega)=\tilde{u}(\omega)\tilde{v}(\omega)\text{ and }\tilde{y}(\omega)=\text{constant }\exp[ik\omega]\ \bar{\tilde{u}}(\omega)\tilde{v}(\omega)$$

so, the theorem's statement is true due to the result of Fourier inverse transform of the above relations.

Note that in some applications the physical signal ***x*** is real vector (all its components $x(n)$'s are real-valued). For this case the roots of $P_a(z)$ will be real numbers or pairwise complex-conjugate, as a result of this fact and theorem 1 different subset of {β_j: j=1, …, N-1, $\beta_j\neq 1/\bar{\beta}_k$ for $j\neq k$} determines different real signal but not unique. In summary, even for real signal the uniqueness in reconstruction cannot be guaranteed without some additional conditions.

4. THE UNIQUENESS CONDITION ON RELATED INTERFERENCE SIGNALS AND RECONSTRUCTION ALGORITHM

In lots of applications the original spectrum {$|\tilde{x}(\omega_j)|$} as well as the interference spectrum {$|\tilde{x}(\omega_j)+\tilde{f}(\omega_j)|$} of the referenced signal ***x***+***f*** with some reference signal ***f*** of finite length is measured. In some situations, e.g., in the experiment of photoconductivity of polymer membrane, ***f***≡{$f(n)$} is completely known. In some other situations, only part of information of ***f*** is known in priori. Such different situations are investigated separately in the following subsections.

4.1 Additional information of completely known interference signal

Theorem 3 Let signal ***f***≡{$f(n)$} of length N be given, then there exist at most two signals ***x***≡{$x(n)$} of length N satisfying the measurement equations for j=1, …, N:

$$|\tilde{x}(\omega_j)|=|\sum_{n\in Z}x(n)\exp(-in\omega_j)| \tag{10}$$

$$|\tilde{y}(\omega_j)|=|\sum_{n\in Z}(x(n)+f(n))\exp(-in\omega_j)| \tag{11}$$

If the spectrum of f is of linear phase qω+ω_0 where q is some integer and ω_0 is some constant frequency, the ***x*** is unique.

Proof Let ***y***=***x***+***f***, $\tilde{x}(\omega_j)=|\tilde{x}(\omega_j)|\exp[i\Phi(\omega_j)]$, $\tilde{f}(\omega_j)=|\tilde{f}(\omega_j)|\exp[i\Psi(\omega_j)]$, the the spectrum of ***y*** is:

$$|\tilde{y}(\omega_j)|^2=|\tilde{x}(\omega_j)+\tilde{f}(\omega_j)|^2=|\tilde{x}(\omega_j)|^2+|\tilde{f}(\omega_j)|^2+2\mathrm{Re}[\tilde{x}(\omega_j)\bar{\tilde{f}}(\omega_j)]$$

$$=|\tilde{x}(\omega_j)|^2+|\tilde{f}(\omega_j)|^2+2|\tilde{x}(\omega_j)||\tilde{f}(\omega_j)|\cos(\Phi(\omega_j)-\Psi(\omega_j)) \quad (12)$$

In equality (16) the data terms $|\tilde{y}(\omega_j)|$, $|\tilde{x}(\omega_j)|$, $|\tilde{f}(\omega_j)|$ and $\Psi(\omega_j)$ are all known so that the phase difference between $\tilde{x}(\omega)$ and $\tilde{f}(\omega)$, i.e., $\Xi(\omega_j)\equiv\Phi(\omega_j)-\Psi(\omega_j)$, can be uniquely determined in $[0, 2\pi)$. Since cosine is an even function, $\Phi(\omega_j)$ has at most two solutions $\Phi_1(\omega_j)$ and $\Phi_2(\omega_j)$:

$$\Phi_1(\omega_j)-\Psi(\omega_j)=-\Phi_2(\omega_j)+\Psi(\omega_j)+2\pi K$$

where K is some integer. Then $\Phi_1(\omega_j)=-\Phi_2(\omega_j)+2\Psi(\omega_j)+2\pi K$.

In case of linear phase $\Psi(\omega)=q\omega+\omega_0$ then:

$$\Phi_1(\omega_j)=-\Phi_2(\omega_j)+2q\omega_j+2\omega_0+2\pi K$$

and the corresponding signals are related by just a simple variant relation:

$$x_1(n)=\overline{x_2(2q-n)}\exp[2i\omega_0]$$

namely, $\boldsymbol{x}$ is unique. This ends the proof.

Sine linear phase reference signals are widely used, the reconstruction algorithm can be established as follows. According to theorem 3, its solution is unique in case of linear phase reference signal.

Algorithm

Parameters: Signal length N, set of sampling points $\{\omega_j: j=1, \ldots, N\}$
Input: Measured spectrum samples $|\tilde{x}(\omega_j)|$ and $|\tilde{y}(\omega_j)|\equiv|\tilde{x}(\omega_j)+\tilde{f}(\omega_j)|$: $j=1, \ldots, N$;
Reference signal $\boldsymbol{f}\equiv\{f(n): n=0, \ldots, N-1\}$

Computation Steps:
(1) Compute $\Xi(\omega_j)=\Phi(\omega_j)-\Psi(\omega_j)\in[0,\pi]$ according to Equation (12);
(2) Compute $\tilde{x}(\omega_j)=|\tilde{x}(\omega_j)|\exp[i(\Phi(\omega_j)+\Xi(\omega_j))]$ for each ω_j;
(3) Compute the signal $\{x(n)\}$ from $\{\tilde{x}(\omega_j)\}$ by fast Fourier inverse transform.

4.2 Additional information of only power spectrum of the interference signal

In some situation the reference signal $\boldsymbol{f}$ is unknown, however, still part of its information can be known such as its power spectrum. The measurement model is:

$$|\tilde{x}(\omega)|=|\textstyle\sum_{n\in Z} x(n)\exp(-in\omega)|,\ |\tilde{f}(\omega)|=|\sum_{n\in Z} f(n)\exp(-in\omega)|,\ |\tilde{y}(\omega)|=|\sum_{n\in Z}(x(n)+f(n))\exp(-in\omega)| \quad (13)$$

with measurements $\{|\tilde{x}(\omega_j)|, |\tilde{f}(\omega_j)|, |\tilde{y}(\omega_j)|: j=1,\ldots,N\}$. A useful uniqueness condition is stated in the following theorem.

Theorem 4 Let $\boldsymbol{x}$ and $\boldsymbol{f}$ be signals of length N_1 and N_2 respectively which satisfy all Equations in (12). If the polynomials $P(z)$ and $Q(z)$ associated with $\{|\tilde{x}(\omega_j)|\}$ and $\{|\tilde{f}(\omega_j)\}$ (defined by (6)) respectively have no common roots, the both $\boldsymbol{x}$ and $\boldsymbol{f}$ are unique.

Remark If signals $\boldsymbol{x}$ and $\boldsymbol{f}$ are independent each other then $P(z)$ and $Q(z)$ obviously have no common root. Therefore, the uniqueness condition in this theorem is very general in practies.

Proof Let $\boldsymbol{x}_1$ and $\boldsymbol{f}_1$ are another pair of signal and reference signal satisfying (13). By theorem 2 there exist finite-length signals $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{z}$ such that:

$$\tilde{x}(\omega)=\tilde{u}(\omega)\ \tilde{v}(\omega),\ \widetilde{x_1}(\omega)=\tilde{u}(\omega)\overline{\tilde{v}(\omega)}\exp[i\rho]$$

$$\tilde{f}(\omega)=\widetilde{w}(\omega),\ \ \tilde{z}(\omega)\exp[-ik\omega] \quad (14)$$

$$\widetilde{f_1}(\omega)=\widetilde{w}(\omega)\overline{\tilde{z}(\omega)}\exp[i\sigma-iq\omega]$$

where ρ and σ are constants in $(-\pi, +\pi]$, k and q are integer constants. Without loss of generality, suppose $u(n)=0$ except for $n=0, \ldots, m_1$; $v(n) = 0$ except for $n=0, \ldots, N_1-1-m_1$; $w(n) = 0$ except for $n=0, \ldots, m_2$; $z(n)=0$ except for $n=0, \ldots, N_2-1-m_2$. For $\boldsymbol{x}_1+\boldsymbol{f}_1$ and $\boldsymbol{x}+\boldsymbol{f}$ there holds:

$$|\tilde{x}(\omega)+\tilde{f}(\omega)|^2=|\tilde{y}(\omega)|^2=|\widetilde{x_1}(\omega)+\widetilde{f_1}(\omega)|^2$$

Substitute (13) into this equality and note that $|\tilde{x}(\omega)|^2=|\widetilde{x_1}(\omega)|^2$, $|\tilde{f}(\omega)|^2=|\widetilde{f_1}(\omega)|^2$, after some straightforward computation there holds:

$$(\exp[ik\omega]\tilde{u}(\omega)\overline{\widetilde{w}(\omega)}-\exp[-i(l\omega+\rho-\sigma)]\overline{\tilde{u}(\omega)}\widetilde{w}(\omega))(\tilde{v}(\omega)\overline{\tilde{z}(\omega)}-\exp[-i((k-q)\omega-\rho+\sigma)]\overline{\tilde{v}(\omega)}\tilde{z}(\omega))=0$$

Hence either

$$\exp[ik\omega]\tilde{u}(\omega)\overline{\widetilde{w}(\omega)}=\exp[-i(q\omega+\rho-\sigma)]\overline{\tilde{u}(\omega)}\widetilde{w}(\omega) \tag{15}$$

or

$$\tilde{v}(\omega)\overline{\tilde{z}(\omega)}=\exp[-i((k-q)\omega-\rho+\sigma)]\overline{\tilde{v}(\omega)}\tilde{z}(\omega) \tag{16}$$

is true.

Suppose (15) is true: in this case the equality (9) and (14) leads to

$$\tilde{u}(\omega)=u(m_1)\prod_{j=1}^{m1}(\exp[-i\omega]-\beta_j),\ \widetilde{w}(\omega)=w(m_2)\prod_{j=1}^{m2}(\exp[-i\omega]-\gamma_j)$$

Under the condition stated in this theorem, $\beta_j\neq\gamma_i$ for any j and i. Substitute these two equations into (15) and after some algebra calculation, it holds:

$$\begin{aligned}&\exp[i\omega m_2]\prod_{j=1}^{m1}(\exp[-i\omega]-\beta_j)\prod_{i=1}^{m2}(\exp[-i\omega]-1/\bar{\gamma}_i)\\ &=C\exp[-i\omega(k+q-m_1)]\prod_{j=1}^{m1}(\exp[-i\omega]-1/\bar{\beta}_j)\prod_{i=1}^{m2}(\exp[-i\omega]-\gamma_i)\end{aligned} \tag{17}$$

where C is some constant independent of ω. Note that both sides are polynomials in $z\equiv\exp[-i\omega]$ and (by compare the degrees of both sides) $m_1=m_2+k+q$. Since both sides have the same set of roots, i.e.

$$\{\beta_j: j=1, \ldots, m_1\}\cup\{1/\bar{\gamma}_i: i=1, \ldots, m_2\}=\{1/\bar{\beta}_j: j=1, \ldots, m_1\}\cup\{\gamma_i: i=1, \ldots, m_2\}$$

Since $\beta_j\neq\gamma_i$ for any j and i, all roots in these sets except those on the unit circle (notated as $\exp[-i\theta_j]$ and $\exp[-i\tau_j]$) can be paired as $\{(\beta_j, 1/\bar{\beta}_j): j=1, \ldots, J_1\}$ and $\{(\gamma_i, 1/\bar{\gamma}_i): i=1, \ldots, J_2\}$ respectively where $J_1\leq m_1$, $J_2\leq m_2$. Hence

$$\begin{aligned}\tilde{u}(\omega)&=u(m_1)\prod_{j=1}^{J1}(\exp[-i\omega]-\beta_j)(\exp[-i\omega]-1/\bar{\beta}_j)\prod_{j=1}^{m1-2J1}(\exp[-i\omega]-\exp[-i\theta_j])\\ \widetilde{w}(\omega)&=w(m_2)\prod_{j=1}^{J2}(\exp[-i\omega]-\gamma_j)(\exp[-i\omega]-1/\bar{\gamma}_j)\prod_{j=1}^{m2-2J2}(\exp[-i\omega]-\exp[-i\tau_j])\end{aligned}$$

Note that

$$\begin{aligned}\widetilde{x_1}(\omega)&=\overline{\tilde{u}(\omega)\tilde{v}(\omega)\exp[i\rho-im_1\omega]}=\exp[i\rho-im_1\omega]\tilde{x}(\omega)\\ \widetilde{f_1}(\omega)&=\overline{\widetilde{w}(\omega)z(\omega)}\exp[i\sigma-iq\omega-im_2\omega]=\overline{\tilde{f}(\omega)}\exp[i\sigma-i(q+k+m_2)\omega]\end{aligned}$$

which implies that $\boldsymbol{x}$ is just a simple variant of $\boldsymbol{x}_1$ and $\boldsymbol{f}$ is also just a simple variant of $\boldsymbol{f}_1$. In a similar way, (16) implies the same result. This ends the proof.

5. CONCLUSIONS AND FURTHER WORKS

In this paper the general structure of the one-dimensional signal with given power spectrum is formulated and on basis of this formulism some conditions are introduced to guarantee the uniqueness of solution in signal reconstruction. Such conditions are about some additional information of the interference signal which is useful and obtainable in the situation where the original spectrum as well as the interference spectrum of the signal with some finite-length reference signal is measured. On basis of this formulism and the uniqueness condition, an algorithm is established for signal construction, which is simple, efficient and the solution is unique in case of linear-phase signals. Some generalized and more practical conditions are also discussed and it is worthwhile to investigate some of them furthermore in future works.

REFERENCES

[1] Eldar, Y. C., [Sampling Theory: Beyond Bandlimited Systems], Cambridge University Press, Cambridge, (2020).

[2] Cong, S. and Zhang, J. J., "Compressive sensing theory, optimization algorithm and application in system state reconstruction," Information and Control 46(3), 267-274 (2019).

[3] Cohen, D. and Eldar, Y. C., "Sub-Nyquist radar systems: Temporal, spectral and spatial compression," IEEE Signal Processing Magazine 15(6), 35-57 (2021).

[4] Kannan, R., Shake, N. and Berman, A., "Survey on nonlinear power spectrum estimation," Signal Processing and Its Applications 25(1), 133-166 (2017).

[5] Duarte, M. F. and Baraniuk, R. G., "Kronecker compressive sensing," IEEE Trans. Image Processing 27(2), 494-504 (2018).

[6] Dai, W., Li, Y. and Zou, J., "Fully decomposable compressive sampling with optimization for multidimensional sparse representation," IEEE Trans. Signal Processing 70(3), 603-616 (2022).

[7] Heckel, R., Morgemshtern, V. and Soltanolkotabi, M., "On super-resolution signal recovery," Information and Inference 15(1), 22-75 (2020).

[8] Dorsch, D. and Rauhut, H., "Refined analysis of sparse phase retrieval," J. Fourier Analysis and Applications 28(6), 1120-1152 (2021).

[9] Aubel, C., Stotz, D. and Boelcskei, H., "Super-resolution from short-time Fourier transform measurements," J. Fourier Analysis and Applications 26(1), 22-35 (2022).

[10] Cohen, R. and Eldar, Y. C., "Sparse Doppler sensing based on nested arrays," IEEE Trans. Ultrasonics, Ferroelectrics and Frequency Control 5(12), 2349-2364 (2021).

[11] Aldroubi, P., Cabrelli, R. and Jaffard, S., "New trends in spectrum analysis: Sparse representation, compressed sensing and multifractal analysis," Proc. Geophysical Signal Processing and Imaging Techniques 65-78 (2023).

[12] Tropp, J. A., Yurtsever, A. and Udell, W., "Practical sketching algorithms for low rank signal approximation," SIAM J. Matrix Anal. Appl. 38(4), 1454-1485 (2018).