

Uniqueness conditions for 1d signal reconstruction from power spectrum with additional data term information

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ABSTRACT

Signal reconstruction from its power spectrum is an important technique widely useful in different application fields. However, uniqueness is generally hard to be guaranteed by usual approaches due to nonlinearity of the power spectrum measurement models. In this paper a very general structure of the 1-dimensional signal with given power spectrum is established and some conditions are introduced to guarantee the uniqueness of solution in signal reconstruction on basis of this formulism. Such conditions are about one or more additional information of the signal component and two algorithms are established for signal construction. One of the algorithms is exact and another one is iterative and approximate, both can reach the real signal under the introduced conditions.

Keywords: Power spectrum, signal reconstruction, non-linear measurement model, uniqueness

1. INTRODUCTION

1.1 Basic problems and related works

Signal reconstruction problem widely appears in different application fields. In this problem the complete signal, i.e., both its magnitudes and phases at a finite number of specified time instants or spatial positions, need to be reconstructed from its power spectrum measurements^{1,2}.

Power spectrum measurement is essentially a non-linear observation model and only magnitude information of signal's Fourier transform is obtained³. As a result, uniqueness is generally hard to be guaranteed by usual approaches due to nonlinearity of the power spectrum measurement model and its information incompleteness. So far, most approaches are based on extending the sampling frequency, e.g., by doubling the number of sampling points, to provide sufficient measurements to guarantee uniqueness^{4,5}. Such methods obvious have lots of limitations in some realistic applications, e.g., it is unfeasible to have adequate sampling density or speed with some acceptable cost, particularly for two or higher dimensional signals⁶. Even for one dimensional situation, e.g., the time sequence signal of finite length, simply enlarging its sample set is in many cases unfeasible in practice⁷⁻⁹. On the other hand, in many typical situations some additional information about the signal itself may be available which can be helpful for reconstructing the real signal to get the unique solution in some conditions. For example, in one dimensional photoelectric lattice and two-dimensional photoelectric membrane experiments not only the absorption spectrum (essentially the power spectrum of the anisotropic photoelectric material's permeability) can be measured but also the value of permeability itself can be measure at some special position¹⁰. In some nonlinear optical experiments, not only the field spectrum at given set of frequencies but also some additional information about the field itself or reference field at some special points can be available^{11,12}. Naturally how to make use of such additional information to develop the reconstruction algorithm is a valuable problem which needs theoretical and practical investigation. However, so far there are only few works in this direction.

1.2 Contributions

In this paper a very general structure of the 1-dimensional signal with given power spectrum is proved and on basis of this formulism some conditions are introduced to guarantee the uniqueness of solution in signal reconstruction. Such conditions are about one or more additional information of the signal component. On basis of this formulism and uniqueness condition, two algorithms are established for signal construction, one of which is exact and the other is iterative and approximate. Both algorithms can reach the real signal under the uniqueness condition.

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2. BASIC PROBLEM, MODELS AND ELEMENTARY PROPERTIES

Notations and Conventions All vectors \mathbf{x} , \mathbf{y} , etc. are column vectors, so \mathbf{x}^T , \mathbf{y}^T , etc. are row vectors. $x(n)$ is the n -th component of vector \mathbf{x} . Z is the set of integers.

2.1 Basic model

The basic model for one-dimensional signal reconstruction is the following observation equation:

$$|\tilde{x}(\omega)|^2 = \left| \sum_{n \in Z} x(n) \exp(-in\omega) \right|^2 \quad (1)$$

In this equation, $\mathbf{x} = \{x(n)\}$ is a sequence of complex-number valued signals with N items, each item $x(n) = |x(n)| \exp[i\varphi(n)]$ where ω is the discretized frequency. The starting instant n_0 and length N of such signal sequence is given, so without loss of generality, let $\{x(n)\}$ be $\{x(0), x(1), \dots, x(N-1)\}$ where $x(0)$ and $x(N-1)$ are non-zero while $x(n) = 0$ for $n < 0$ or $n \geq N$. All unessential sampling parameters and scale factors are neglected.

The basic problem is: reconstruct signal $\mathbf{x} = \{x(n)\}$ according to observations $\{|\tilde{x}(\omega_j)|\}$ which are a collection of data at discretized frequencies $\{\omega_j\}$.

It is emphasized that the observation $|\tilde{x}(\omega_j)|$ has only magnitude information in frequency domain, however, what is to be constructed is the complete signal $x(n) = |x(n)| \exp[i\varphi(n)]$ with information of both magnitude $|x(n)|$ and phase $\varphi(n)$.

In the following analysis the auto-correlation sequence plays an important role which definition is:

$$a(n) = \sum_{k \in Z} \tilde{x}(k) x(n+k) \quad (2)$$

The auto-correlation sequence $\mathbf{a} = \{a(n)\}$ of signal $\{x(0), x(1), \dots, x(N-1)\}$ is a sequence of size $2N-1$ where $a(n) = 0$ for $|n| > N$. The Fourier spectrum of \mathbf{a} and the Fourier spectrum of \mathbf{x} are related by

$$\tilde{a}(\omega) = \sum_{n, k \in Z} \tilde{x}(k) x(n+k) \exp(-in\omega) = |\tilde{x}(\omega)|^2 \quad (3)$$

Note that the signal \mathbf{x} of size N has $2N$ to-be-determined real-number valued variables: N magnitudes $|x(n)|$ and N phases $\varphi(n)$. As a result totally $2N$ observations $|\tilde{x}(\omega_j)|$ in model (1) are needed to reconstruct the complete signal \mathbf{x} . By the well-known interpolation formula, such observations can determine the continuous power spectrum $\tilde{a}(\omega) = |\tilde{x}(\omega)|^2$ of the auto-correlation sequence \mathbf{a} , i.e.,

$$\tilde{a}(\omega) = \sum_{j=1}^{2N} |\tilde{x}(\omega_j)|^2 \prod_{i=1, i \neq j}^{2N} \frac{\sin(\omega - \omega_i)/2}{\sin(\omega_j - \omega_i)/2} \quad (4)$$

where $\omega_i - \omega_j \neq 2m\pi$ for any $m \in Z$. The auto-correlation sequence \mathbf{a} can be completely determined by Fourier transform of (4). Due to non-linearity of the model, however, signal \mathbf{x} itself cannot be completely determined by this approach. For example, it's easy to verify that all signals $\{\exp(-i\alpha)x(n)\}$, $\{x(n-m)\}$ and $\{x(-n)\}$ have the same auto-correlation sequence as signal $\{x(n)\}$. Such signals are called hereafter the simple variant of the original signal $\{x(n)\}$, which have the same physical significance of \mathbf{x} in practical applications. With this consideration, the signal reconstruction problem can be re-stated more exactly by:

Given observations $\{|\tilde{x}(\omega_j)|\}$ which are a collection of data at discretized frequencies $\{\omega_j\}$, reconstruct signal $\mathbf{x} = \{x(n)\}$ which is unique in the sense that all different solutions are within the simple variant category of the exact signal \mathbf{x} .

As a result of the following analysis, the uniqueness cannot be simply reached by $2N$ observations, therefore some additional observations or constraints are necessary. Our analysis in the following also shows under which additional conditions the uniqueness can be really guaranteed.

2.2 Some related models

In practice some measurement models in different forms frequently appear, e.g., the electric conductivity measurement model used in the research of polymer electronic devices is:

$$|\tilde{x}(\omega)|^2 = \left| \sum_{n \in Z} x(n) K(\omega, n) \right|^2 \quad (5)$$

where the kernel function

$$K(\omega, n) = (2\pi b)^{-1/2} \exp(-i\pi/4) \exp[(i/2b)(an^2 - 2\omega n + d\omega^2)]$$

By straightforward calculation, it holds that

$$|\tilde{x}(\omega)|^2 = \left| \sum_{n \in Z} x(n) \exp[(ian^2/2b] \exp(-i\omega n) \right|^2$$

i.e., model (5) can be reduced to model (1) if $x(n) \exp[(ian^2/2b]$ rather than $x(n)$ is regarded as the signal. In its application domain a and b are given parameters so signal reconstruction problem of model (5) is equivalent to the same problem of model (1).

Many other observation models in applications have the relations with model (1) similar as the above, hence model (1) is typical and the following analysis is focused on it.

3. GENERAL FORMALISM AND MAIN RESULT ON UNIQUENESS

Non-uniqueness is the most essential obstacle for signal reconstruction on basis of model (1). In this section the complete form of signal \mathbf{x} is derived from measurement (equation (1)). As a result, the most important impact factors for the uniqueness can be determined and furthermore the uniqueness condition can be established on basis of this analysis.

3.1 General formalism and main result

For a given sequence $\mathbf{a} = \{a(n): n=1-N, \dots, N-1\}$ of length N , define an associated polynomial of degree $2N-2$:

$$P_a(z) = \sum_{n=0}^{2N-2} a(n-N+1)z^n \quad (6)$$

For the auto-correlation sequence \mathbf{a} of signal \mathbf{x} it always holds that

$$\tilde{a}(\omega) = |\tilde{x}(\omega)|^2 = \exp[-i(N-1)\omega] P_a(\exp(-i\omega)) \quad (7)$$

By equation (2): $\overline{a(-n)} = a(n)$ so it's true that

$$\overline{z^{2(N-1)} P_a(1/\bar{z})} = \overline{P_a(z)}$$

As a result, if γ is a non-zero root of $P_a(z)$ then so is $1/\bar{\gamma}$, i.e., the roots of $P_a(z)$ can be paired as $(\gamma, 1/\bar{\gamma})$ which are distributed inside and outside the unit circle symmetrically.

Equation (2) implies $\overline{a(1-N)} = a(N-1) = x(0)x(N-1) \neq 0$ (otherwise at least one of $x(0), x(N-1) = 0$ which conflicts the assumption for sequence \mathbf{x}), so $P_a(0) \neq 0$. This implies:

$$P_a(z) = a(N-1) \prod_{j=1}^{N-1} (z - 1/\bar{\gamma}_j)(z - \gamma_j) \quad (8)$$

By (7) and $\tilde{a}(\omega) = |\tilde{x}(\omega)|^2$ there holds that

$$\begin{aligned} \tilde{a}(\omega) &= |P_a(\exp(-i\omega))| \\ &= |a(N-1)| \prod_{j=1}^{N-1} |(\exp(-i\omega) - 1/\bar{\gamma}_j)(\exp(-i\omega) - \gamma_j)| = |a(N-1)| \prod_{j=1}^{N-1} |\gamma_j|^{-1} \prod_{j=1}^{N-1} |\exp(-i\omega) - \gamma_j|^2 \end{aligned}$$

For real-valued frequency ω then

$$\tilde{x}(\omega) = |a(N-1)|^{1/2} \prod_{j=1}^{N-1} |\beta_j|^{-1/2} \prod_{j=1}^{N-1} (\exp(-i\omega) - \beta_j) \quad (9)$$

where $\{\beta_j: j=1, \dots, N-1 \text{ and } \beta_j \neq 1/\bar{\beta}_k \text{ for any } j \neq k\}$ is a subset of roots.

Since signal \mathbf{x} is determined by its spectrum $\tilde{x}(\omega)$, equation (9) shows the general formalism of the solution to Equation (1). Nevertheless it is not unique because different subset of $\{\beta_j\}$ of cardinality $N-1$ determines different solutions.

The above argument essentially proves the theorem which is the main result in this section:

Theorem 1 Any signal \mathbf{x} implied by equation (1), up to its simple variant, must has the spectrum in form of (9) where $\{\beta_j: j=1, \dots, N-1 \text{ and } \beta_j \neq 1/\bar{\beta}_k \text{ for any } j \neq k\}$ is the subset of roots of polynomial $P_a(z)$ defined in (6).

This theorem is the foundation to develop the signal reconstruction algorithm. An interesting corollary of this theorem is that if $P_a(z)$ associated with the measurement data only has roots on the unit circle then the constructed signal is unique (up to simple variants). Such situation cannot occur in general, but the more is the number of roots on the unit circle on the complex plane, the less is the number of different solutions. Let $L = \{(\gamma_j, 1/\bar{\gamma}_j): \gamma_j \neq 1/\bar{\gamma}_j\}$ be the number of pairs not distributed along the unit circle, m_j be the multiplicity of the j -th pair $(\gamma_j, 1/\bar{\gamma}_j)$, then it is easy to verify that the number of signals implied by Equation (1) is at most:

$$(1/2)\prod_{j=1}^L(1 + m_j)$$

In applications theorem 1 has another useful equivalent form which can be stated as the following.

Theorem 2 Let \mathbf{x} and \mathbf{y} be two signals of length N and both satisfying Equation (1), then there exist two signals \mathbf{u} and \mathbf{v} of finite lengths such that

$$\mathbf{x}=\mathbf{u} \# \mathbf{v} \text{ and } \mathbf{y}=\{\exp[i\alpha]\overline{u(-n)}\} \# \{v(n-m)\}$$

where $\#$ is the convolution operator, α is some real-number constant and m is an integer constant.

The proof of this theorem is straightforward. Since by theorem 1 signal both \mathbf{x} and \mathbf{y} can be represented by (9) but with different set of roots $\{\beta_j: j=1, \dots, N-1\}$ and $\{\gamma_j: j=1, \dots, N-1\}$, then

$$\tilde{\mathbf{x}}(\omega)=\text{constant } \tilde{\mathbf{y}}(\omega)\prod_{j=1}^{N-1} \frac{\exp(-i\omega)-\gamma_j}{\exp(-i\omega)-\beta_j}$$

Let $\{\beta_1, \dots, \beta_m\}$ and $\{\gamma_1, \dots, \gamma_m\}$ are the largest ono-intersection subset of roots where $|\gamma_j|\leq 1, j=1, \dots, t$ and $|\gamma_j|>1, j=t+1, \dots, m$. Since the distribution of $P_a(z)$'s roots are symmetric relative to the unit circle, there should exist exactly t roots $\beta_j=1/\overline{\gamma_j}, j\leq t$ and other $m-t$ roots $\beta_j=1/\overline{\gamma_j}, j\geq t+1$, therefore:

$$\frac{\overline{\tilde{\mathbf{x}}(\omega)}}{\overline{\tilde{\mathbf{y}}(\omega)}}=\text{constant}\prod_{j=1}^m \frac{\exp(-i\omega)-\gamma_j}{\exp(-i\omega)-1/\overline{\gamma_j}}=\text{constant } \exp[im\omega]\prod_{j=1}^m \frac{\exp(-i\omega)-\gamma_j}{\exp(+i\omega)-\overline{\gamma_j}}=\text{constant } \exp[im\omega] \tilde{\mathbf{u}}(\omega)/\tilde{\mathbf{u}}(\omega)$$

where $\tilde{\mathbf{u}}(\omega)\equiv\prod_{j=1}^m(\exp(-i\omega)-\gamma_j)$. Equivalently:

$$\overline{\tilde{\mathbf{x}}(\omega)}/\tilde{\mathbf{u}}(\omega)=\text{constant } \exp[im\omega]\tilde{\mathbf{y}}(\omega)/\tilde{\mathbf{u}}(\omega)$$

Let $\tilde{\mathbf{v}}(\omega)\equiv\overline{\tilde{\mathbf{x}}(\omega)}/\tilde{\mathbf{u}}(\omega)$ then

$$\tilde{\mathbf{x}}(\omega)=\tilde{\mathbf{u}}(\omega)\tilde{\mathbf{v}}(\omega) \text{ and } \tilde{\mathbf{y}}(\omega)=\text{constant } \exp[ik\omega] \tilde{\mathbf{u}}(\omega)\tilde{\mathbf{v}}(\omega)$$

so, the theorem's statement is true due to the result of Fourier inverse transform of the above relations.

Note that in some applications the physical signal \mathbf{x} is real vector (all its components $x(n)$'s are real-valued). For this case the roots of $P_a(z)$ will be real numbers or pairwise complex-conjugate, as a result of this fact and theorem 1 different subset of $\{\beta_j: j=1, \dots, N-1, \beta_j\neq 1/\overline{\beta_k} \text{ for } j\neq k\}$ determines different real signal but not unique. In summary, even for real signal the uniqueness in reconstruction cannot be guaranteed without some additional conditions.

4. THE UNIQUENESS CONDITIONS AND APPLICATION TO ALGORITHMS

Theorem 1 in last section provides a useful starting point to establish signal reconstruction algorithm as well as shows the reason for non-uniqueness. Of course there are different conditions suitable in different applications. In this section a very useful and compact condition is presented with only one additional data (recall that "uniqueness" is in the sense of "up to the simple variant).

Theorem 3 Suppose in model (1) $2N$ equations for $\{|\tilde{\mathbf{x}}(\omega_j)|\}$ are given together with one additional data $C=|x(N-1)|$ which is the magnitude of the $N-1$ 'th component of signal \mathbf{x} , let $B\equiv\{\beta_1, \dots, \beta_{N-1}: \beta_j$'s are the roots of $P_a(z)$ satisfying (10)\}:

$$|C|^2=|a(N-1)|\prod_{j=1}^{N-1} |\beta_j|^{-1} \tag{10}$$

If the condition

$$\prod_{\beta_j \in A} |\beta_j|^{-1} \neq 1 \tag{11}$$

holds for any subset A of B then there exists only one signal \mathbf{x} satisfying model (1) with data $\{|\tilde{\mathbf{x}}(\omega_j)|\}$ and $C=|x(N-1)|$.

Proof Let \mathbf{x} be the signal satisfying (1), then by (9) up to a constant factor $\exp(-i\theta)$:

$$x(N-1)=\text{the coefficient of the term } \exp(-i\omega(N-1)) \text{ in the expression of } \tilde{\mathbf{x}}(\omega)=|a(N-1)|^{1/2}\prod_{j=1}^{N-1} |\beta_j|^{-1/2}$$

Substitute the above expression into $C=|x(N-1)|$ and make square operation then (10) is obtained, i.e., the set B exists.

Let y be another signal also satisfying (1). According to theorem 1 there is a corresponding set of $N-1$ roots $\{\gamma_j\}$ satisfying (10):

$$|C|^2 = |a(N-1)| \prod_{j=1}^{N-1} |\gamma_j|^{-1}$$

Such $N-1$ roots γ_j 's are symmetric with β_j 's relative to the unit circle, i.e., there exists $L: 1 \leq L \leq N-1$ such that (after some re-numbering of the roots):

$$\gamma_j = 1/\bar{\beta}_j, j \leq L; \gamma_j = \beta_j, j \geq 1+L$$

so

$$\prod_{j=1}^{N-1} |\gamma_j| = \prod_{j=1}^L |\gamma_j| \prod_{j=1+L}^{N-1} |\gamma_j| = \prod_{j=1}^L |\beta_j|^{-1} \prod_{j=1+L}^{N-1} |\beta_j|$$

On the other hand:

$$\prod_{j=1}^{N-1} |\gamma_j| = |a(N-1)|/|C|^2 = \prod_{j=1}^{N-1} |\beta_j|$$

so, by combining the above two equalities it is obtained that:

$$\prod_{j=1}^{N-1} |\beta_j| = \prod_{j=1}^L |\beta_j|^{-1} \prod_{j=1+L}^{N-1} |\beta_j|$$

namely

$$\prod_{j=1}^L |\beta_j|^2 = 1 \tag{12}$$

which implies non-uniqueness of x , as a result there must exist some subset A of B such that $A = \{\beta_1, \dots, \beta_L\}$ satisfies (12). Inversely, if (11) holds for any subset of B then x is unique. This ends the proof.

From the above analysis it's easy to see that a and y can at most differentiate from each other by a constant factor $\exp(-i\theta)$, i.e., the phase differences $\varphi(n) - \varphi(m)$ between any two components $x(n)$ and $x(m)$ are definite. This shows that such uniqueness condition is practical in applications.

An extension of theorem 3 is to consider any additional signal data $C = |x(k)|$ with some $k: 1 \leq k \leq N-1$. Such extension is true and y the same arguments as the above, the following theorem can be proved:

Theorem 4 Suppose in model (1) $2N$ equations for $\{\tilde{x}(\omega_j)\}$ are given together with one additional data $C = |x(k)|$ which is the magnitude of the k 'th component of signal x , let $B = \{\beta_1, \dots, \beta_{N-1}\}$: β_j 's are the roots of $P_a(z)$ satisfying (12)}:

$$\Omega_B = \sum_{1 \leq j_1 < \dots < j_{N-k-1} \leq 1} \beta_{j_1} \dots \beta_{j_{N-k-1}}, |C|^2 = |\Omega_B|^2 |a(N-1)| \prod_{j=1}^{N-1} |\beta_j|^{-1}$$

If the condition

$$\prod_{\beta_j \in A} |\beta_j|^{-1} \neq \frac{\sum_{j \in A} \beta_j + \sum_{j \notin A} \beta_j}{\sum_{j \in A} 1/\bar{\beta}_j + \sum_{j \notin A} \beta_j} \tag{13}$$

holds for any subset A of B then there exists only one signal x satisfying model (1) with data $\{\tilde{x}(\omega_j)\}$ and $C = |x(k)|$.

On basis of the above theoretical results, some signal reconstruction algorithms can be established with the unique solution. For simplicity, such algorithms are only presented under the condition $C = |x(N-1)|$.

Algorithm I

Parameters: signal length N , set of frequency points $\{\omega_j: j=1, \dots, 2N\}$ for measurement.

Input: measurements, $\tilde{h}_j = |\tilde{x}(\omega_j)|, j=1, \dots, 2N$ and additional information $C = |x(N-1)|$.

Computation steps:

- (1) Compute the polynomial $P_a(z)$ from $\{|\tilde{x}(\omega_j)|^2\}$.
- (2) Compute all the roots of $P_a(z)$.
- (3) Find a subset $B = \{\beta_1, \dots, \beta_{N-1}: \beta_j \neq 1/\bar{\beta}_k \text{ for } j \neq k\}$ to satisfy condition (10).
- (4) Find any subset A of B which is invalid against condition (11).
- (5) If A does not exist, then the roots in B determine the exact spectrum of signal x and such x is computed by Fourier inverse-transform.

If A exists, then the signal cannot be reconstructed from the given input and parameters.

Note that Step (1) can be performed by any highly efficient interpolation algorithm. Step (2) can be performed by any highly efficient algorithms in numerical linear algebra and only those roots inside (or outside) the unit circle needs to be computed. In step (3), $D \equiv \log(|C|^2/a(N-1))$ and $\sigma_j \equiv \log|\beta_j|$ for each $|\beta_j| \neq 1$ can be computed at first and let \bar{A} be the set of all $\sigma_j < 0$ (total number of non-positive σ_j 's is $N-1$). Let $\Sigma(\bar{A})$ be the sum of all members in \bar{A} , if $\Sigma(\bar{A}) < D$ the negative member $-\sigma_m$ with the largest magnitude in \bar{B} is replaced with $+\sigma_m$ and re-compute $\Sigma(\bar{A})$ for the new \bar{B} ; if $\Sigma(\bar{A}) > D$ then the negative member $-\sigma_k$ with the secondary largest magnitude in \bar{B} is replaced with $+\sigma_k$ and re-compute $\Sigma(\bar{A})$ for the new \bar{B} . Since \bar{A} is of finite cardinality and the condition (10) holds, the computation will terminate with the desired set B of roots after at most $N-1$ steps

In some applications more additional data may be available, e.g., $C_1 = |x(k_1)|$ and $C_2 = |x(k_2)|$. Although this situation is obviously more helpful to find the unique solution, a straightforward extension of the above algorithm will be too complicated. On the other hand, a more heuristic approach is feasible that a solution is computed by the above algorithm with only one additional data, $C_1 = |x(k_1)|$ say, at first and then furthermore computation is performed with the remaining data, e.g., $C_2 = |x(k_2)|$.

Another approach for signal reconstruction is to take such problem as a satisfiability problem. Given power spectrum $\{\tilde{h}(\omega_j)\}$ and the additional information $|x(N-1)| = C$, define the domains A and B in signal space as:

$$A \equiv \{z \in \mathbb{C}^N : |z(N-1)| = C\}, B \equiv \{z \in \mathbb{C}^N : |\tilde{z}(\omega_j)| = \tilde{h}(\omega_j), j=1, \dots, 2N\}$$

then the signal reconstruction problem is equivalent to find a vector $\mathbf{x} \in A \cap B$. In order to solve such problem an efficient iterative algorithm can be established, i.e., to compute an iteration sequence $\mathbf{x}_{t+1} = P_A P_B \mathbf{x}_t$ from some initial estimate \mathbf{x}_0 where $P_A(\mathbf{u})$ and $P_B(\mathbf{u})$ are operators which map \mathbf{u} to the point in A and B with the shortest distance to \mathbf{u} . Since P_A and P_B are compressive operators, the convergence is guaranteed in the iteration and the limit is the real signal under the uniqueness condition.

According to the above idea, the second algorithm is as the following.

Algorithm II

Parameters: iteration control parameter T and ε , signal length N, set of frequencies $\{\omega_j: j=1, \dots, 2N\}$ for measurement.

Input: measurements, $\tilde{h}_j \equiv |\tilde{x}(\omega_j)|, j=1, \dots, 2N$ and additional information $C = |x(N-1)|$.

Computation steps:

(1) Initialize \mathbf{x}_0 for $t=0$.

(2) At each ω_j compute $\tilde{x}_t(\omega_j) = \sum_{n=0}^{N-1} x_t(n) \exp(-in\omega_j)$ for the current $\mathbf{x}_t \equiv (x_t(0), x_t(1), \dots, x_t(N-1))$.

(3) If $|\tilde{x}_t(\omega_j)| > \varepsilon$ then $\tilde{g}_t(\omega_j) = \tilde{h}_j \tilde{x}_t(\omega_j) / |\tilde{x}_t(\omega_j)|$ otherwise:

$$\tilde{g}_t(\omega_j) = \tilde{h}_j, v_t(n) = (1/N) \sum_{j=1}^{2N} \tilde{g}_t(\omega_j) \exp(in\omega_j), n=0, \dots, N-1, \mathbf{v}_t \equiv (v_t(0), v_t(1), \dots, v_t(N-1))$$

(4) Compute

$$\mathbf{x}_{t+1} \equiv (x_{t+1}(0), x_{t+1}(1), \dots, x_{t+1}(N-1)) = (v_t(0), v_t(1), \dots, v_t(N-2), C \exp[i \text{Arg } v_t(N-1)]) \text{ with } t=t+1$$

(5) If $t < T$ and $|\mathbf{v}_t - \mathbf{v}_{t-1}| > \varepsilon$ then goto step (2) otherwise output \mathbf{x}_t .

For different additional information of $C = |x(m)|$, algorithm II only has to be slightly modified to compute

$$\mathbf{x}_{t+1} = (v_t(0), v_t(1), \dots, v_t(m-1), C \exp[i \text{Arg } v_t(N-1)], v_t(m+1), \dots, v_t(N-1))$$

in step 2 instead of significant modification like algorithm I. Also, in this algorithm Fourier transform dominates the workload and in step 2 computations for $\tilde{x}_t(\omega_j)$ and $\tilde{g}_t(\omega_j)$ can be easily accelerated by parallelism.

5. CONCLUSIONS

In this paper the general structure of the one-dimensional signal with given power spectrum is formulated and on basis of this formulism some conditions are introduced to guarantee the uniqueness of solution in signal reconstruction. Such conditions are about one or more additional information of the signal and two algorithms are developed for signal construction. One of the algorithms is exact and the other is iterative, both can be efficiently implemented and reach the real signal under the introduced conditions.

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