Chapter 1

Imaging laws of geometrical optics[ii]

§1. Construction of a ray refracted by a spherical surface
Let M (Fig. 1) be the center of the refracting sphere of radius r and refractive index n', and the ambient medium have the refractive index n. To find the refracted ray from the incident ray LE, we insert, according to the elegant method of construction of Weyerstraß, two auxiliary circles 1 and 2 with radii

$$r_1 = \frac{n'}{n} r$$

and

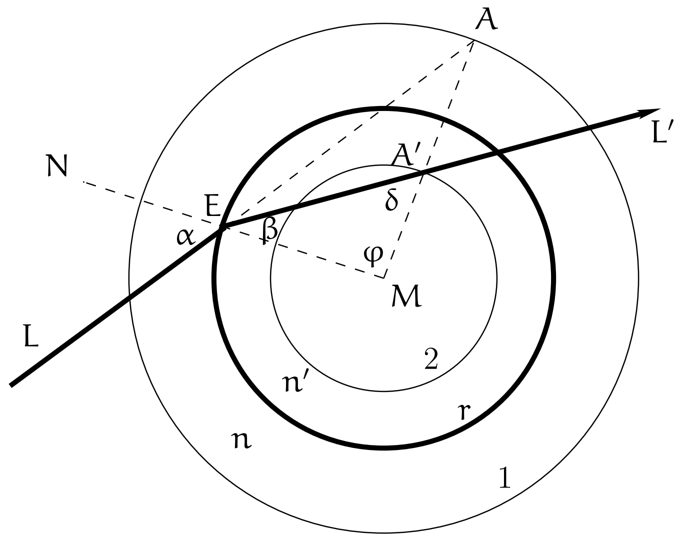
$$r_2 = \frac{n}{n'} r \,,$$

extend ray LE until it intersects auxiliary circle 1 at A, and connect E with point A' where line AM and auxiliary circle 2 intersect. Line $EA'L'$ is the refracted ray associated with LE.

From the similarity of triangles EAM and $EA'M$, it follows that

$$\angle MEA = \angle EA'M$$ [iii]

Figure 1



or

$$\alpha = \delta \, ;$$

further,

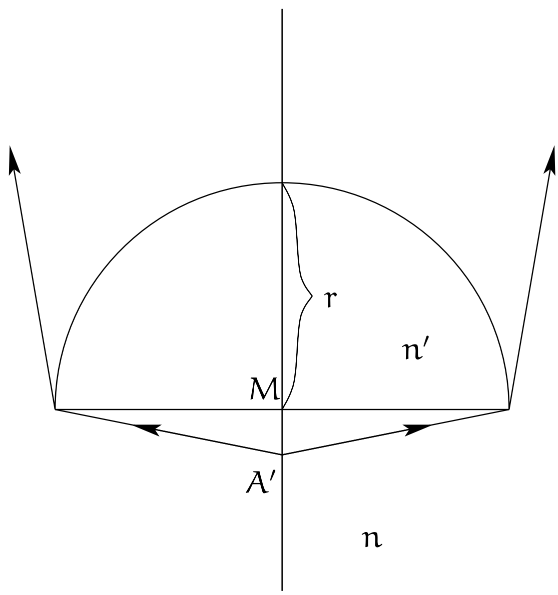
$$\frac{\sin \delta}{\sin \beta} = \frac{EM}{A'M} = \frac{n'}{n} \, ;$$

therefore,

$$\frac{\sin \alpha}{\sin \beta} = \frac{n'}{n} \, .$$

It follows immediately from this construction that all incident rays aiming toward A go through point A' after refraction. It follows therefore from the law of reciprocity that all outgoing rays from point A' in medium n' go through A after refraction, if they are extended backward. We want to designate these outstanding points A and A' as "aberration-free" points of a refracting spherical surface because the spherical aberration for them is zero. This "aberration-free" pair of points plays a major role in the construction of microscope objectives. One employs, e.g., a semispherical glass lens as the front lens in the apochromat (see Fig. 2).[iv] If one uses homogeneous immersion and brings the object to be imaged to distance $A'M = \frac{n}{n'}r$ of aberration-free point A', the divergence of near 180° is considerably reduced, without the occurrence of spherical aberration.

Figure 2

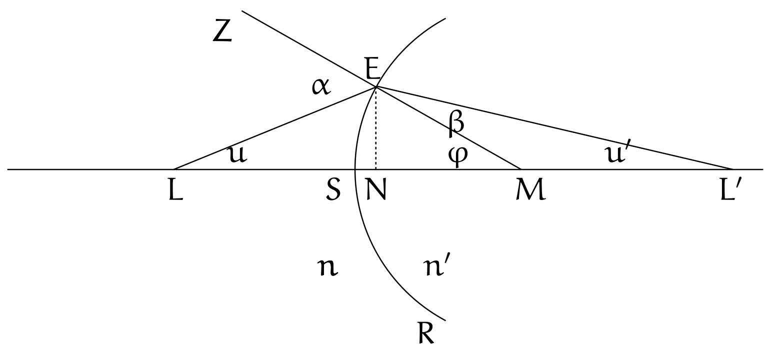


In order to learn more about the path of a ray bundle that is not coming from aberration-free points, we follow its path in analytical ways.

§2. Imaging of an arbitrary luminous axial point

Let M (Fig. 3) be the center of refracting spherical surface RSE that separates media n and n', onto which luminous point L sends rays. The unrefracted ray LSM passing through the spherical surface is

Figure 3



designated as the central ray and taken as the axis of the refractive system. If EL' is the refracted ray associated with LE, then

$$n \sin\alpha = n' \sin\beta \ .$$

According to the figure, the following holds:

$$\frac{\sin\alpha}{\sin u} = \frac{LM}{ME} \quad \text{and} \quad \frac{\sin\beta}{\sin u'} = \frac{L'M}{ME} \ ;$$

therefore,

$$\frac{LM}{L'M} = \frac{n' \sin u'}{n \sin u} \ .$$

Further, we have

$$\frac{\sin u'}{\sin u} = \frac{LE}{L'E} \ ,$$

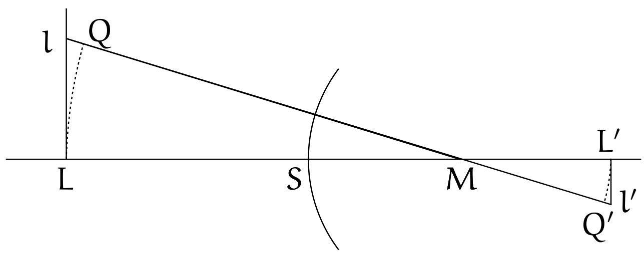
and therefore,
$$\frac{LM}{L'M} \cdot \frac{L'E}{LE} = \frac{n'}{n};$$
the ratio $LM//L'M$ *is in general dependent on* u. We shall show that it becomes independent of u only if u and u' are small, that is, if we image using paraxial pencils (null rays).

Let us drop a vertical line EN onto the axis. Then,
$$LE = \frac{LN}{\cos u} = \frac{LS + SN}{\cos u} \text{ or } LE = \frac{LS + EM(1 - \cos\varphi)}{\cos u}.$$
Analogously,
$$L'E = \frac{L'S - EM(1 - \cos\varphi)}{\cos u'}.$$
If u, u', and therefore φ are so small that one can set $\cos u$, $\cos u'$, and $\cos\varphi = 1$, thus $LE = LS$ and $L'E = L'S$; then,
$$\frac{LM}{L'M} \cdot \frac{L'S}{LS} = \frac{n'}{n}. \tag{1}$$
Since $\frac{L'S}{LS}$ is completely independent of u, one therefore obtains the following theorem: *homocentric null rays remain homocentric after refraction.*[v]

§3. Imaging of luminous objects

If a second point Q (Fig. 4) is present besides the luminous axial point L, then what is valid for L in relation to LM is also valid for Q in relation to the neighboring axis QM. If one restricts oneself to point Q very close to axis LM, one sees that all object points lying on arc LQ with radius LM are imaged point-to-point onto the arc $L'Q'$ with radius ML'. Since one can, with the introduced restrictions, use instead of arcs LQ and $L'Q'$ their projections Ll and $L'l'$, we have the following theorem: *small surfaces perpendicular to the axis are imaged point-to-point as surfaces perpendicular to this axis.*

Figure 4



Since the conjugate points lie on the line going through the center of the sphere, we have

$$\frac{lL}{l'L'} = \frac{LM}{L'M} \quad \text{or} \quad \frac{y}{y'} = \frac{s-r}{s'-r}\,,$$

where lines y and y' are taken to be positive or negative depending on whether they lie above or below the axis. Since it was shown that[vi]

$$\frac{s-r}{s'-r} = \frac{n'}{n} \cdot \frac{s}{s'}\,,$$

therefore

$$\frac{y'}{y} = \frac{n}{n'} \cdot \frac{s'}{s}\,. \tag{2}$$

If one designates $\frac{y'}{y} = \beta$ as the "*lateral magnification*," the following theorem is valid: *the lateral magnification is constant for conjugate planar pairs, but varies from pair to pair.*

From the figure,[vii] it is clear that

$$\frac{\tan u}{\tan(-u')} = \frac{+s'}{-s} \quad \text{or} \quad \frac{\tan u}{\tan u'} = \frac{s'}{s}\,,$$

where u and u' are to be evaluated as positive if the associated ray or its extension is rotated about L and L' in a clockwise fashion in order to reach the axis. By combining the last equation with Eq. 2, one gets

$$y'n'\tan u' = yn\tan u\,. \tag{3}$$

If one designates $\frac{\tan u'}{\tan u} = \gamma$ as "*angular magnification*," then

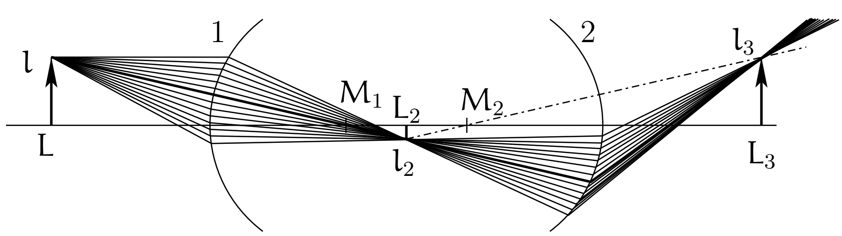
$$\beta\gamma = \frac{n}{n'}\,. \tag{4}$$

That is, "*The product of the lateral magnification and angular magnification is constant*" (law of Lagrange).

§4. Imaging by a centered system of refracting spherical surfaces

In a centered system, the centers of the refracting spherical surfaces all lie on a straight line, which we choose as the axis. Image L_2l_2 of object Ll (Fig. 5) produced by the first spherical surface can itself be interpreted as the object that generates image L_3l_3. Image point L_2 distinguishes itself from a self-luminous object in the same location in that its outgoing ray bundle does not fill completely the aperture of spherical surface 2. Nevertheless, l_2 will be imaged as a point that is l_3. Since this is also valid for every refracting spherical surface

Figure 5



that follows, we have the theorem: *the object space is imaged point-to-point in the image space. Planes perpendicular to the axis in the object space correspond point-to-point to the planes perpendicular to the axis in the image space*.

If one applies the Lagrange relation to each refracting surface in the system successively, one obtains the Lagrange–Helmholtz relation

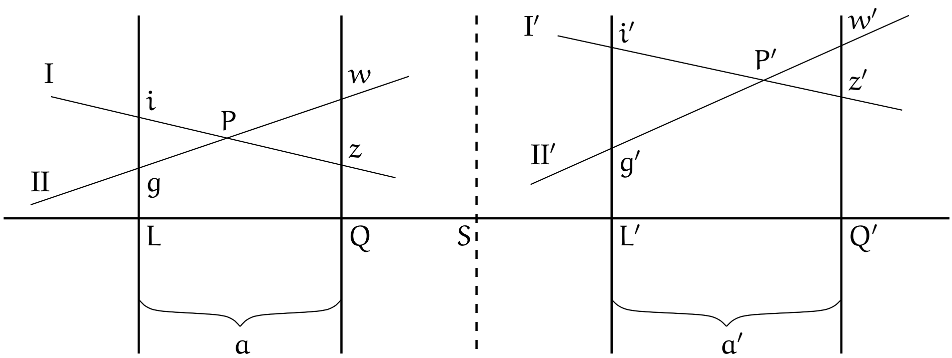
$$\left.\begin{aligned} \beta \cdot \gamma &= \frac{n}{n'} \\ \text{or} \qquad y' n' \tan u' &= y n \tan u \end{aligned}\right\}, \tag{5}$$

where β and γ now denote the lateral magnification and angular magnification with respect to the *entire system*, and n and n' are refractive indices of the front (object) and back (image) media.

§5. Imaging equations according to Abbe

In Fig. 6, let there be conjugate pairs of planes L and L' as well as Q and Q', and the associated lateral magnifications be given by

Figure 6

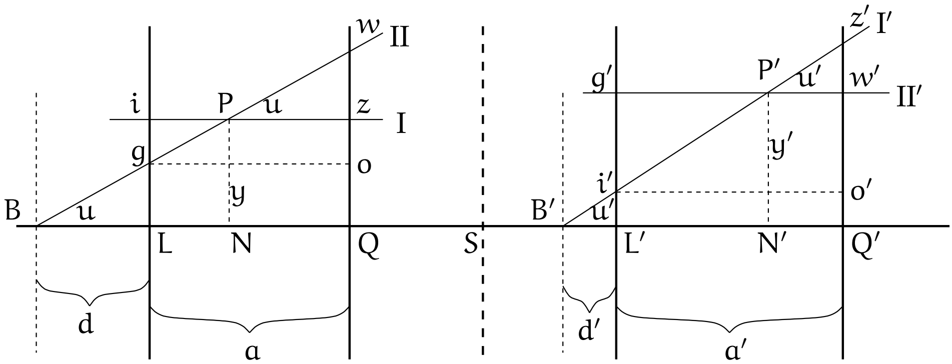


$$\frac{y_1'}{y_1} = \nu_1 \text{ and } \frac{y_2'}{y_2} = \nu_2 \; ;$$

overall imaging is thus determined. Let, e.g., a general ray I intersect object planes at i and z; one finds then, using values ν_1 and ν_2, the conjugate points i' and z' and with them the conjugate ray I′.[viii] Construct ray II analogously. Since every point P of the object space can be considered as an intersection of two rays that cut through planes L and Q, one can therefore, for every object point, find its conjugate image point P'.[ix]

To derive the imaging equations, we consider the special case (Fig. 7) of letting I run parallel to the axis while so directing II that its conjugate ray II′ runs parallel to the axis in the *image space*. In this case, for ray I, we have

Figure 7



$$y_1 = y_2 = y$$

$$\frac{y_2'}{y_1'} = \frac{\nu_2}{\nu_1} \, ,$$

and for ray II,

$$y'_1 = y'_2 = y'$$
$$\frac{y_2}{y_1} = \frac{\nu_1}{\nu_2} ;$$

therefore,

$$\frac{B'L'}{i'L'} = \frac{L'Q'}{Q'z' - i'L'} \quad \text{or} \quad \frac{d'}{y'_1} = \frac{a'}{y'_2 - y'_1} ,$$

and so

$$d' = a' \frac{\nu_1}{\nu_2 - \nu_1} .$$

Analogously, we have

$$d = -a \frac{\nu_2}{\nu_2 - \nu_1} .$$

These equations tell us that distance d' is independent of y and distance d is independent of y'. We therefore have the theorem: *all rays parallel to the axis in the object space meet at an axial point in the image space (back focal point* B'*), and all outgoing rays from a definite axial point in the object space (front focal point* B*) travel parallel to the axis in the image space.*

From Fig. 7, we have

$$\tan u' = \frac{y'_2 - y'_1}{a'} = \frac{y}{a'} \left(\frac{y'_2}{y} - \frac{y'_1}{y} \right) ,$$

and therefore

$$\frac{y}{\tan u'} = \frac{a'}{\nu_2 - \nu_1} = F' , \tag{6}$$

where F' is a constant of the optical system and is defined (after Gauß) as the "*focal length*" of the image space. Analogously,

$$\frac{y'}{\tan u} = \frac{a \nu_1 \nu_2}{(\nu 1 - \nu 2)} = F , \tag{7}$$

which is defined as the focal length of the object space.

We determine the location of P using a coordinate system whose z-axis coincides with the axis of the optical system and whose origin coincides with the front focal point B. We obtain the position of P′ using B′ as the origin. Let the positive sense of coordinates z and z' follow the direction of light propagation. Then,

$$\frac{y}{z} = \tan u \quad \text{and} \quad \frac{y'}{z'} = \tan u' \,.$$

If one combines these with the defining equations of focal lengths, one gets, finally,

$$\left.\begin{aligned} z \cdot z' &= F \cdot F' \\ \frac{y'}{y} &= \frac{F}{z} = \frac{z'}{F'} \end{aligned}\right\} \,. \tag{8}$$

The imaging equations in this form were first established by Abbe.[x]

§6. Imaging by wide-angle ray bundles (sine condition)

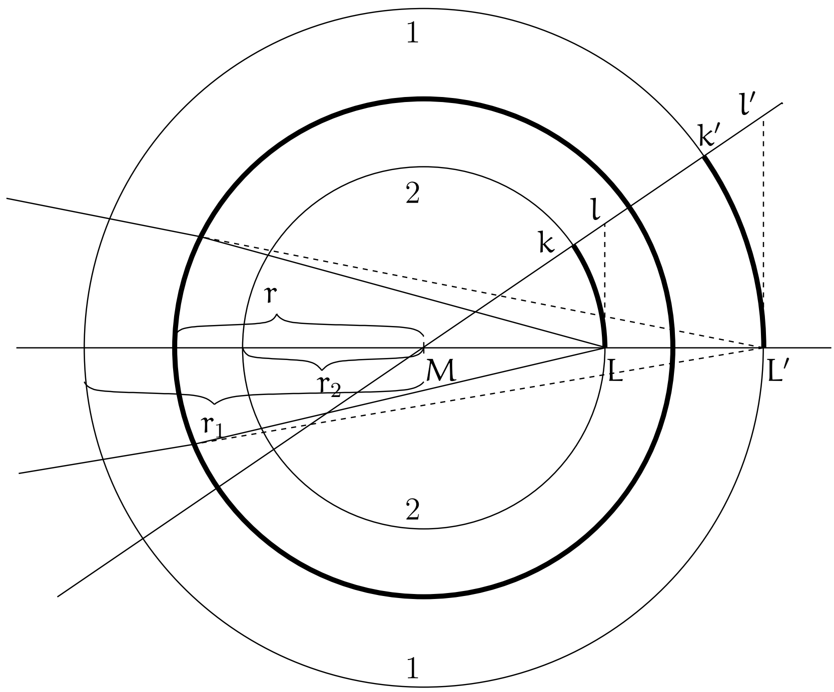
(a) *A refracting spherical surface.* Point-to-point imaging using null rays has no meaning in microscopic imaging, since it is necessary here, for reasons to be explained later, to bring *wide-angle* bundles of rays to union. The question is whether and under what conditions point-to-point imaging is possible by wide-angle bundles of rays in general.

As we have seen, this goal can be reached by a single refracting spherical surface for only a single pair of conjugate axial points.[xi] For this aberration-free pair of points, the relationship derived in § 2 is strictly valid:

$$\frac{LM}{L'M} = \frac{n' \sin u'}{n \sin u} = \text{const;}$$

i.e., the length of convergence[xii] $L'M$ is independent of the opening angle u of the ray bundle.[xiii]

Figure 8



As one can see from Fig. 8, this is also valid for points k and k' with respect to the neighboring axis Mkk' for wide-angle ray bundles in general:

$$\frac{kM}{k'M} = \frac{n' \sin u'}{n \sin u} = \text{const},$$

where the constant has the same value as above. Therefore, arbitrarily large arc Lk of circle 2 with radius $r_2 = \frac{n}{n'} r$ can be imaged by general wide-angle ray bundles point-to-point and similar in perspective with respect to M, so that Lk is associated with the arc situated on the circle with radius $r_1 = \frac{n'}{n} r$ by

$$L'k' = Lk \cdot \frac{L'M}{LM} .$$

If we limit ourselves to very small objects Lk, we can set

$$\frac{L'k'}{Lk} = \frac{L'l'}{Ll} = \beta$$

and get

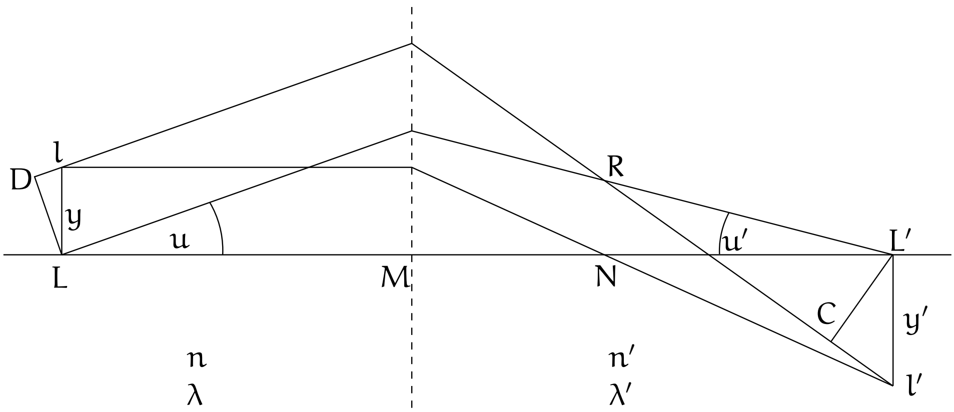
$$\frac{\sin u'}{\sin u} = \frac{n}{n'} \cdot \frac{1}{\beta} . \tag{9}$$

This is the condition under which a perpendicular-to-axis surface element at aberration-free point L is imaged as another perpendicular-to-axis surface element at the conjugate point L' point-to-point and in similarity by arbitrarily wide-angle ray bundles. It is called the "*sine condition*," and the conjugate aberration-free pair of points, for which this condition is satisfied, are called the "*aplanatic points*" of the refracting surface.

(b) *A centered system.* We now ask ourselves, which condition must be satisfied so that in a centered system of refracting spherical surfaces, a perpendicular-to-axis surface element Ll (Fig. 9) can be imaged to form another perpendicular-to-axis surface element $L'l'$ point-to-point and in similarity by wide-angle ray bundles in general. *All* rays coming from L should be refracted toward L', and rays from point l should be refracted toward the conjugate point l'.

The condition that all rays coming from axial point L are reunited at L' is identical therefore to stating that the system be

Figure 9



free of spherical aberration. In addition, should the rays coming from point l be reunited at the conjugate point l', a further condition must be satisfied such that the system for conjugate points l and l' with respect to the neighboring axis lMl' is free of spherical aberration. To find this condition, we track, according to the simple derivation by John Hockins,[1] two parallel rays originating from L and l in the object space, whose intersection in the image space is at R. The parallel-to-axis ray from l intersects the axis in the image space at N. We now draw perpendicular lines $L'C$ and LD. As a result of the absence of spherical aberration for the pair of points L and L', the following is valid for the optical lengths:

$$\overline{LRL'} = \overline{LNL'} \,.$$

But since

$$\overline{lRl'} = \overline{lNl'} \,,$$

[1] *Journ. Roy. Microscop. Soc.* 1884, Ser. 2, **4**, 337.

the following must be valid as well:

$$\overline{lRl'} - \overline{LRL'} = \overline{lNl'} - \overline{LNL'} = \overline{lN} + \overline{Nl'} - \overline{LN} - \overline{NL'}\,;$$

since Ll is the wave front of the parallel-to-axis rays that meet at N, it is moreover valid that

$$\overline{LN} = \overline{lN}\,,$$

and therefore

$$\overline{lR} + \overline{Rl'} - \overline{LR} - \overline{RL'} = \overline{Nl'} - \overline{NL'}$$

or

$$\begin{aligned} (\overline{lR} - \overline{LR}) + (\overline{Rl'} - \overline{RL'}) &= \overline{Nl'} - \overline{NL'} \\ -\overline{Dl} + \overline{Cl'} &= \overline{Nl'} - \overline{NL'} \\ \overline{Cl'} &= (\overline{Nl'} - \overline{NL'}) + \overline{Dl}\,. \end{aligned}$$

If line segment $L'l'$ is small to the first order, the difference $\overline{Nl'} - \overline{NL'}$ is small to the second order and is therefore negligible compared to line segment $\overline{Dl}$.[xiv] We therefore obtain

$$\overline{Cl'} = \overline{Dl}\,;$$

or, if we transition to the equivalent line segments in vacuum, we have

$$n'L'l' \cdot \sin(-u') = nLl \cdot \sin u\,,$$

for u' is negative according to the prior agreement. Now,

$$\frac{Ll}{L'l'} = \frac{y}{-y'} = -\frac{1}{\beta}\,;$$

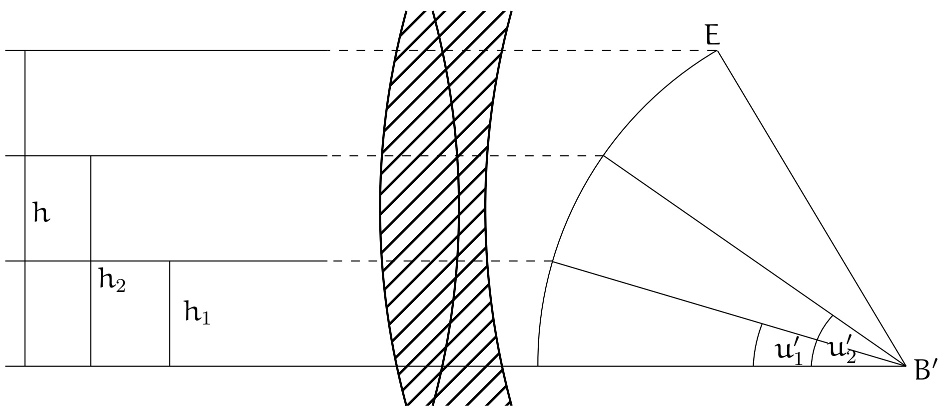
therefore we have, finally,

$$\frac{\sin u'}{\sin u} = \frac{n}{n'}\frac{1}{\beta}\,. \tag{10}$$

This *sine condition* is identical purely dioptrically in that *the various zones of the system of the object project a magnified image of the same ratio* β *at the same position* (the point of convergence of the null zone).

The sine condition takes on a very simple form if either the object or the image point lies at infinity. Then the sine condition goes from (see Fig. 10)[xv]

Figure 10



$$\frac{\sin u_1}{\sin u_1'} = \frac{\sin u_2}{\sin u_2'} = \text{const}$$

over to

$$\frac{h_1}{\sin u_1'} = \frac{h_2}{\sin u_2'} = \text{const}\,,$$

since the quotient $\frac{\sin u_1}{\sin u_2}$ under unbounded growth of the object distance approaches the value h_1/h_2 in the limit. Therefore,

$$\frac{h}{\sin u'} = \text{const} ;$$

since for very small values of u' we have

$$\frac{h}{\sin u'} = \frac{h}{\tan u'} = F' ,$$

the sine condition in this special case reads

$$\frac{h}{\sin u'} = F' \tag{11a}$$

or, if the image is at infinity,

$$\frac{h'}{\sin u} = F . \tag{11b}$$

One can see from Fig. 10 that

$$\sin u' = \frac{h}{EB'} ;$$

therefore, the following must be valid:

$$EB' = F' ,$$

that is, *the intersections of the extended parallel-to-axis incoming rays with their conjugate image rays must lie on a spherical surface having the back focal point* B' *as the center and the focal length of the system* F' *as the radius.*